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ASYMPTOTICALLY EXACT LOCAL DISCONTINUOUS GALERKIN ERROR ESTIMATES FOR THE LINEARIZED KORTEWEG-DE VRIES EQUATION IN ONE SPACE DIMENSION

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Abstract. We present and analyze *a posteriori* error estimates for the local discontinuous Galerkin (LDG) method for the linearized Korteweg-de Vries (KdV) equation in one space dimension. These estimates are computationally simple and are obtained by solving a local steady problem with no boundary condition on each element. We extend the work of Hufford and Xing [*J. Comput. Appl. Math.*, 255 (2014), pp. 441-455] to prove new superconvergence results towards particular projections of the exact solutions for the two auxiliary variables in the LDG method that approximate the first and second derivatives of the solution. The order of convergence is proved to be $k + 3/2$, when polynomials of total degree not exceeding k are used. These results allow us to prove that the significant parts of the spatial discretization errors for the LDG solution and its spatial derivatives (up to second order) are proportional to $(k + 1)$-degree Radau polynomials. We use these results to construct asymptotically exact *a posteriori* error estimates and prove that, for smooth solutions, these *a posteriori* LDG error estimates for the solution and its spatial derivatives, at a fixed time t, converge to the true errors at $\mathcal{O}(h^{k+3/2})$ rate in the L^2-norm. Finally, we prove that the global effectivity indices, for the solution and its spatial derivatives, converge to unity at $\mathcal{O}(h^{1/2})$ rate. Numerical results are presented to validate the theory.

Key words. Local discontinuous Galerkin method, KdV, superconvergence, Radau points, *a posteriori* error estimates.

1. Introduction

The famous nonlinear Korteweg-de Vries (KdV) equation

$$u_t + \alpha u_x + \gamma u u_x + \beta u_{xxx} = 0,$$

with constants α, β, and γ, is derived by Korteweg and de Vries in 1895. It describes the propagation of waves in a variety of nonlinear dispersive media. The KdV equation is a generic equation for the study of weakly nonlinear long waves and arises in many physical situations, such as surface water waves and plasma waves. It has been shown that the KdV equation describes a large class of solitons observed in various situations: acoustic waves on a crystal lattice, plasma waves, hydrodynamics internal or surface waves, elastic surface waves, and waves in optical fibers (see *e.g.*, [27]).

In this paper we develop and analyze an implicit residual-based *a posteriori* error estimates of the spatial errors for the semi-discrete local discontinuous Galerkin (LDG) method applied to the linearized KdV equation

$$u_t + \alpha u_x + \beta u_{xxx} = 0, \quad x \in [a, b],\ t \in [0, T], \tag{1a}$$

subject to the initial and periodic boundary conditions

$$u(x, 0) = u_0(x), \quad x \in [a, b], \tag{1b}$$

$$u(a, t) = u(b, t), \quad u_x(a, t) = u_x(b, t), \quad u_{xx}(a, t) = u_{xx}(b, t), \quad t \in [0, T]. \tag{1c}$$

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We would like to emphasize that the assumption of periodic boundary conditions is for simplicity only and is not essential. In our analysis we select $u_0(x)$ such that the exact solution $u(x,t)$ is a smooth function on $[a,b] \times [0,T]$.

The LDG method we discuss in this paper is an extension of the discontinuous Galerkin (DG) method aimed at solving partial differential equations containing higher than first-order spatial derivatives. The DG method is a class of finite element methods, using discontinuous, piecewise polynomials as the numerical solution and the test functions. It was first developed by Reed and Hill [31] for solving hyperbolic conservation laws containing only first-order spatial derivatives in 1973. Consult [25] and the references cited therein for a detailed discussion of the history of DG method and a list of important citations on the DG method and its applications. The LDG method for solving convection-diffusion problems was first introduced by Cockburn and Shu in [26]. They further studied the stability and error estimates for the LDG method. Castillo *et al.* [19] presented the first *a priori* error analysis for the LDG method for a model elliptic problem. They considered arbitrary meshes with hanging nodes and elements of various shapes and studied general numerical fluxes. They showed that, for smooth solutions, the L^2 errors in ∇u and in u are of order k and $k+1/2$, respectively, when polynomials of total degree not exceeding k are used. Cockburn *et al.* [24] presented a superconvergence result for the LDG method for a model elliptic problem on Cartesian grids. They identified a special numerical flux for which the L^2-norms of the gradient and the potential are of orders $k+1/2$ and $k+1$, respectively, when tensor product polynomials of degree at most k are used.

Yan and Shu [35] developed the first LDG method for solving KdV type equations in one and two space dimensions. They proved L^2 stability and a cell entropy inequality for the square entropy for a class of nonlinear KdV equations in both one and multiple space dimensions. They also proved an optimal error estimate for the linear cases in the one-dimensional case. In [33], Xu and Shu proved L^2 error estimates for the semi-discrete LDG methods for the fully nonlinear KdV equation with smooth solution. The order of convergence is proved to be $k+1/2$, when k-degree piecewise polynomials with $k \geq 1$ are used. Later, Xu and Shu [34] proved optimal L^2 error estimates of the semi-discrete LDG methods for solving linear higher-order wave equations including the linearized KdV equation. More recently, Hufford and Xing [30] studied the superconvergence property of the LDG method for solving the linearized KdV equation. They selected a special projection of the initial condition and proved that the LDG solution is $\mathcal{O}(h^{k+3/2})$ super close to a particular projection of the exact solution, when the upwind flux is used for the convection term and the alternating flux is used for the dispersive term.

A posteriori error estimates lie in the heart of every adaptive finite element algorithm for differential equations. They are used to assess the quality of numerical solutions and guide the adaptive enrichment process where elements having high errors are enriched by h-refinement and/or p-refinement while elements with small errors are h- and/or p-coarsened. Furthermore, error estimates are used to stop the adaptive refinement process. For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [6]. Several *a posteriori* DG error estimates are known for hyperbolic [22, 23, 28] and diffusive [29, 32] problems. Adjerid and Baccouch [3, 12, 10] investigated the global convergence of the implicit residual-based *a posteriori* error estimates of Adjerid *et al.* [5]. They proved that these *a posteriori* error estimates converge to the true spatial error in the L^2-norm