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A TWO-GRID FINITE VOLUME ELEMENT METHOD FOR A NONLINEAR PARABOLIC PROBLEM

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Abstract. A two-grid algorithm is presented and discussed for a finite volume element method to a nonlinear parabolic equation in a convex polygonal domain. The two-grid algorithm consists of solving a small nonlinear system on a coarse-grid space with grid size H and then solving a resulting linear system on a fine-grid space with grid size h. Error estimates are derived with the H^1-norm $O(h + H^2)$ which shows that the two-grid algorithm achieves asymptotically optimal approximation as long as the mesh sizes satisfy $h = O(H^2)$. Numerical examples are presented to validate the usefulness and efficiency of the method.

Key words. Two-grid, finite volume element method, nonlinear parabolic equation, error estimates.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$, and consider the initial-boundary value problem

$$
\begin{cases} u_t - \nabla \cdot (A(u)\nabla u) = f(x,t), & x \in \Omega, 0 < t \le T, \\ u(x,t) = 0, & x \in \partial\Omega, 0 < t \le T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}
$$

where u_t denotes $\frac{\partial u}{\partial t}$, $x = (x_1, x_2)$, $f(x,t)$ is a given real-valued function on Ω. We assume that the coefficient $A(u)$ is sufficiently smooth such that there exist constants C_i $(i = 1, 2, 3)$ satisfying

$$
0 < C_1 \le A(u) \le C_2, \quad |A(u)_t| \le C_3, \quad \forall u \in C(\Omega \times [0,T]), \tag{2}
$$

and the Lipschitz continuous condition, $\forall u, v \in C(\Omega \times [0,T])$,

$$
|A(u) - A(v)| \le L|u - v|, \quad |A(u)_t - A(v)_t| \le L|u - v|, \tag{3}
$$

with L a positive constant and $A(u)_t = \frac{\partial}{\partial t} A(u)$.

It is also assumed that the functions f, u_0 have enough regularity and they satisfy appropriate compatibility conditions so that the initial-boundary value problem (1) has a unique solution satisfying the regularity results as demanded by our subsequent analysis [1].

We shall study a two-grid algorithm of a nonlinear parabolic equation by using finite volume element method (FVEM). The FVEM is a class of important numerical methods for solving differential equations. It has been widely used in many engineering fields, such as computational fluid mechanics, groundwater hydrology, heat and mass transfer and petroleum engineering, reservoir simulations. Perhaps the most important and attractive property of the FVEM is that it possesses local conservation laws (mass, momentum and energy) which is crucial in many applications. Many researchers have studied this method for linear and nonlinear problems.

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We can refer to [2-11] for general presentation of this method and references therein for details.

On the other hand, two-grid method is a discretization technique for nonlinear equations based on two grids of different sizes. The main idea is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems, and then use it as the initial guess on the fine grid. This method involves a nonlinear solve on the coarse grid with grid size H and a linear solve on the fine grid with grid size $h < H$, respectively. Two-grid method was firstly introduced by Xu [12, 13] for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial differential equations. Later on, two-grid method was further investigated by many authors. We can refer to [14] for finite difference method and to [15, 16, 17, 18, 19, 20, 21] for finite element and mixed finite element method. For finite volume element method, there are also many literatures [22, 23, 24, 25, 26, 27, 28]. For the nonlinear parabolic problem (1), Dawson and Wheeler [14, 15] have constructed the two-grid method by using finite difference method and mixed finite element method. Chen and Liu [21] have studied the two-grid piecewise linear finite element method. Recently, In [24, 25] the two-grid finite volume element method was studied for the semilinear parabolic problem with a nonlinear reaction term, but with a linear diffusion term. Zhang et al. [27, 28] have considered the two-grid finite volume element method for circumcenter based control volumes, with suboptimal estimates in L^2 and H^1 norms for a nonlinear parabolic equation.

However, as far as we know, there is no convergence analysis of the two-grid FVEM based on barycenter control volumes for the nonlinear parabolic problem (1). In this paper, we consider the two-grid FVEM for barycenter based control volumes for the nonlinear parabolic problem (1). The two-grid FVEM is based on two conforming piecewise linear finite element spaces S_H and S_h. Where S_H is the coarse grid with grid size H and S_h is the fine grid with grid size h respectively. With the proposed techniques, solving the nonlinear system on the fine-grid space is reduced to solving a linear system on the fine-grid space and a nonlinear system on a much smaller space. The work for solving the nonlinear system on the coarse-grid space is relatively negligible since $\dim S_H \ll \dim S_h$. This means that solving such a nonlinear problem is not much more difficult than solving one linear problem. A remarkable fact about this simple approach is, as shown in [12], that the coarse grid can be quite coarse and still maintain a good accuracy approximation. The main results in this paper are the error estimates for the considered single-grid and two-grid methods in the Sobolev H^1 norm. To get the estimates, we used standard results from the finite volume element convergence analysis which is based upon viewing the finite volume element method as a perturbation of finite element method.

The rest of this paper is organized as follows. In Section 2 we describe the finite volume element scheme for the nonlinear parabolic problem (1). Section 3 contains the error estimates for the semidiscrete finite volume element method. In Section 4 we construct the two-grid finite volume element algorithm and prove its optimal error estimates in the H^1 norm. Finally in Section 5 we give the numerical examples to validate the theoretical results.

2. Finite volume element method

We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ [29] with $1 \leq p \leq \infty$ consisting of functions that have generalized derivatives of order s in the space