INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 12, Number 2, Pages 230–253

AN ADAPTIVE LINEAR TIME STEPPING ALGORITHM FOR SECOND-ORDER LINEAR EVOLUTION PROBLEMS

JUNJIANG LAI AND JIANGUO HUANG*

Abstract. In this paper, we propose and analyze a linear time stepping finite element method for abstract second order linear evolution problems. For such methods, we derive optimal order a posteriori error estimates and sharp a posteriori nodal error estimates using the energy approach and the duality argument. Based on these estimates, we further design an adaptive time stepping strategy for the previous discretization in time. Several numerical experiments are provided to show the reliability and efficiency of the a-posteriori error estimates and to assess the effectiveness of the proposed adaptive time stepping method.

Key words. A posteriori error analysis, adaptive algorithm, linear finite element, evolution problems.

1. Introduction

Adaptive time stepping methods are very important in developing efficient algorithms for solving evolution problems arising from fluid dynamics, epitaxial growth and many other applied sciences (cf. [15, 17, 22, 26, 27]). Such methods are able to adopt feasible time steps for time discretization, largely reducing the computational cost for getting numerical solutions with desired accuracy. The strategy for choosing time steps adaptively is very tricky and problem oriented, and one typical approach corresponds to the construction of an a-posteriori error estimator for the problem under discussion (cf. [6, 10]). Hence, a posteriori error analysis is very useful in constructing efficient adaptive time stepping methods.

In the past decade, people have witnessed rapid and sophisticated progresses in a posteriori error analysis for abstract first order evolution problems (cf. [1–4, 14]). One of the key points of the analysis in these references relies on a higher order reconstruction of the approximate solution, such that the reconstructed function is globally continuous as well as a quasi-projector of the approximate solution in some sense (cf. (3.4) in [3] and (2.7) in [4]). In light of this reconstructed function or its further modification, the optimal order a posteriori error analysis was established by the energy method. The a-posteriori superconvergence estimates for the error at the nodes for Galerkin and Runge-Kutta methods were also derived in [4].

However, to our knowledge, there are few results about a posteriori error analysis for abstract second order evolution problems (even in linear case), which frequently occur in structural analysis (cf. [9, 11, 12]). In the reference [5], Bernardi and Süli proposed a fully discrete scheme for the linear wave equation. The discretization for time derivatives is conducted by the backward Euler scheme with variable steps. In order to derive a posteriori error analysis, they first extended

Received by the editors February 27, 2014, and in revised form, November 10, 2014.
2000 *Mathematics Subject Classification.* 65M60, 65M15.
*Corresponding author.
The first author is supported by NSFC (Grant No. 11101199) and Program for New Century Excellent Talents in Fujian Province University (Grant No. JA12260). The second author is supported by NSFC (Grant No. 11171219) and E-Institutes of Shanghai Municipal Education Commission (Grant No. E03004).

the approximate solutions (the discrete displacement and velocity fields together) defined on time nodes to the whole time interval by linear interpolation, and then obtained a first-order system of error equations. Based on this system and a very technical derivation, they derived a posteriori error bound in time. Moreover, they also discussed a posterior error analysis in time and space together. In the reference [14], as an application of their a posteriori error analysis for abstract first-order problems, Makridakis and Nochetto obtained some a-posteriori error estimates for the second-order linear wave equation by reformatting it as a system of first-order equations; for details, see Corollaries 3.13 and 3.14 in [14]. In the reference [7], Georgoulis, Lakkis and Makridakis proposed a numerical method to solve a linear wave equation by discretizing the time derivatives via the central difference method and carrying out spatial discretization via the finite element method. They used a novel space-time reconstruction of the approximate solution (cf. Definition 4.1 in [7]) and some other techniques to achieve a posteriori $L^\infty(L^2)$-error bound for this method. Based on the time stepping method used in [11,12], Huang, Lai and Tang proposed in [8] a discrete method for solving abstract second-order linear evolution problems and developed a posteriori error analysis systematically.

For the method in [8], we conduct the discretization of time derivatives by means of quadratic continuous discontinuous Galerkin (DG) method, so it has high accuracy in approximation. The computational overheads of the method involve numerical solution of a linear system with the linear operator having a 2 by 2 block structure. However, as is well-known, it is a challenging issue to develop fast solvers for such a system.

In order to balance the accuracy and computational cost, in this paper we propose and analyze linear time stepping methods for abstract second-order evolution problems. For this purpose, we first give the problem we are solving. For any real number $T > 0$, we want to find $u : [0, T] \to D(A)$ satisfying

$$\begin{cases} u''(t) + Au(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0, \\ u'(0) = v_0, \end{cases} \tag{1}$$

where $(\cdot)'$ and $(\cdot)''$ denote respectively the first and second order derivatives in time, A is a positive definite, self-adjoint, linear operator on a Hilbert space $(H, \langle\cdot,\, \cdot\rangle)$ with domain $D(A)$ dense in H, and f is a function from $[0, T]$ into H. Throughout this paper, we assume that

$$u_0,\, v_0 \in D(A), \quad f \in L^2(0,\, T;\, H). \tag{2}$$

To simplify the presentation, we refer the reader to the monograph [25] for details about the standard notation corresponding to the above problem.

To discretize problem (1), we use a standard finite element approach for handling second-order evolution problems, see, e.g., [11–13]. We first define a non-uniform subdivision for the time interval $I := (0, T)$:

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

and use the notations

$$J_n = (t_{n-1},\, t_n], \quad k_n = t_n - t_{n-1}, \quad 1 \le n \le N.$$