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A HYBRIDIZED WEAK GALERKIN FINITE ELEMENT METHOD FOR THE BIHARMONIC EQUATION

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Abstract. This paper presents a hybridized formulation for the weak Galerkin finite element method for the biharmonic equation based on the discrete weak Hessian recently proposed by the authors. The hybridized weak Galerkin scheme is based on the use of a Lagrange multiplier defined on the element interfaces. The Lagrange multiplier is verified to provide a numerical approximation for certain derivatives of the exact solution. An error estimate of optimal order is established for the numerical approximations arising from the hybridized weak Galerkin finite element method. The paper also derives a computational algorithm (Schur complement) by eliminating all the unknowns associated with the interior variables on each element, yielding a significantly reduced system of linear equations for unknowns on the element interfaces.

Key words. Weak Galerkin, hybridized weak Galerkin, finite element methods, weak Hessian, biharmonic problems.

1. Introduction

In this paper, we are concerned with new developments of weak Galerkin finite element methods for partial differential equations. In particular, we shall employ the usual hybridization technique [7, 1, 6] to the weak Galerkin finite element method for the biharmonic equations proposed and analyzed in [13].

For simplicity, we consider the following biharmonic equation with Dirichlet and Neumann boundary conditions: Find $u \in H^2(\Omega)$ such that $u = \xi$ and $\frac{\partial u}{\partial \mathbf{n}} = \eta$ on the boundary of the domain, and satisfying

$$(1) \qquad (1-\nu)\int_\Omega \nabla^2 u : \nabla^2 v d\Omega + \nu \int_\Omega \Delta u \Delta v d\Omega = \int_\Omega f v d\Omega, \qquad \forall v \in H_0^2(\Omega).$$

Here Ω is an open bounded domain in the Euclidean space $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$, $\nabla^2 v$ is the Hessian tensor of v, Δv is the Laplacian of v, and $\nu \in [0, \frac{1}{2}]$ is the Poisson ratio of the plate. For simplicity, we consider the case of $\nu = 0$ so that the weak form is reduced to

$$(2) \qquad \int_\Omega \nabla^2 u : \nabla^2 v d\Omega = \int_\Omega f v d\Omega, \qquad \forall v \in H_0^2(\Omega).$$

The weak Galerkin method is a finite element technique that approximates differential operators (e.g., gradient, divergence, curl, Laplacian, Hessian, etc) as distributions. The method has been successfully applied to several classes of partial differential equations, such as the second order elliptic equation [15, 8, 14], the Stokes equation [16], the Maxwell's equations [10], and the biharmonic equation [9, 13]. For example, in [13], a weak Galerkin finite element method was developed for the biharmonic equation (1) by using polynomials of degree $P_k/P_{k-2}/P_{k-2}$ for

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any $k \geq 2$, where P_k was used to approximate the function u on each element and P_{k-2} was employed to approximate the trace of u and ∇u on the boundary of each element. The objective of this paper is to exploit the use of hybridization techniques in the weak Galerkin methods that shall further relax the connection of the finite element functions among elements.

Hybridization is a useful technique in the finite element methods. The key to hybridization is to identify a Lagrange multiplier which can be used to relax certain constrains (e.g., continuity) imposed on the finite element function across element interfaces. Hybridization has been employed in the mixed finite element methods to yield hybridized mixed finite element formulations suitable for efficient implementation in practical computation [1, 3, 4, 5, 7, 11, 12]. The idea of hybridization was also used in discontinuous Galerkin methods [2] for deriving hybridized discontinuous Galerkin (HDG) finite element methods [6].

We shall show in this paper that hybridization is a natural approach for the weak Galerkin finite element methods. For illustrative purpose, we demonstrate how hybridization can be accomplished for the weak Galerkin finite element scheme of [13]. We shall also establish a theoretical foundation to address critical issues such as stability and convergence for the hybridized weak Galerkin (HWG) finite element method. The hybridized weak Galerkin is further used as a tool to derive a Schur complement problem for variables defined on element boundaries. Therefore, the Schur complement involves the solution of a linear system with significantly less number of unknowns than the original WG or HWG formulation. We believe the hybridization technique is widely applicable in weak Galerkin family for various partial differential equations, and would like to encourage interested readers to conduct some independent study along this direction.

Throughout the paper, C appearing in different places denotes different constant. Let T be a polygonal or polyhedral domain with boundary ∂T. Denote by $(\cdot,\cdot)_T$ and $\langle\cdot,\cdot\rangle_{\partial T}$ the usual inner products in $L^2(T)$ and $L^2(\partial T)$. $\|\cdot\|_{m,T}$ denotes the norm in the Sobolev space $H^m(T)$. $|\cdot|_{m,T}$ stands for the semi-norm of order m. For simplicity, $\|\cdot\|_{m,\Omega}$, $|\cdot|_{m,\Omega}$, $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_\Omega$ are denoted as $\|\cdot\|_m$, $|\cdot|_m$, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, respectively. $\|\cdot\|_{0,T}$ and $\|\cdot\|_{0,\partial T}$ are simply denoted by $\|\cdot\|_T$ and $\|\cdot\|_{\partial T}$, respectively.

The paper is organized as follows. In Section 2, we introduce a weak Hessian and a discrete weak Hessian by using polynomial approximations. In Section 3, we present a HWG finite element algorithm for the biharmonic problem (1). In Section 4, we verify all the stability conditions in Brezzi's theorem [3] for the HWG scheme. In Section 5, we derive an error equation for the HWG approximation. In Section 6, we establish an error estimate of optimal order for the numerical approximation. Finally in Section 7, we present a Schur complement by eliminating all the variables on the element, yielding a system of linear equations with significantly reduced number of unknowns defined on the element boundary.

2. Weak Hessian and Discrete Weak Hessian

By a weak function on T, we mean a triplet $v = \{v_0, v_b, \mathbf{v}_g\}$ such that $v_0 \in L^2(T)$, $v_b \in L^2(\partial T)$ and $\mathbf{v}_g \in [L^2(\partial T)]^d$. Let $\mathcal{W}(T)$ be the space of all weak functions on T; i.e.,

$$\mathcal{W}(T) = \{v = \{v_0, v_b, \mathbf{v}_g\} : v_0 \in L^2(T), v_b \in L^2(\partial T), \mathbf{v}_g \in [L^2(\partial T)]^d\}. \tag{3}$$

For classical functions, the Hessian is a square matrix of second order partial derivatives if they all exist. If $f(x_1, \cdots, x_d)$ stands for the function, then the