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THE CLIQUE AND COCLIQUE NUMBERS' BOUNDS BASED ON THE H-EIGENVALUES OF UNIFORM HYPERGRAPHS

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Abstract. In this paper, some inequality relations between the Laplacian/signless Laplacian H-eigenvalues and the clique/coclique numbers of uniform hypergraphs are presented. For a connected uniform hypergraph, some tight lower bounds on the largest Laplacian H^+-eigenvalue and signless Laplacian H-eigenvalue related to the clique/coclique numbers are given. And some upper and lower bounds on the clique/coclique numbers related to the largest Laplacian/signless Laplacian H-eigenvalues are obtained. Also some bounds on the sum of the largest/smallest adjacency/Laplacian/signless Laplacian H-eigenvalues of a hypergraph and its complement hypergraph are showed. All these bounds are consistent with what we have known when k is equal to 2.

Key words. H-eigenvalue, clique, coclique, hypergraph, tensor, signless Laplacian, Laplacian, adjacency.

1. Introduction

In the current combinatorics and graph theory associative literatures, a growing number of them studied hypergraphs and their applications in various fields [1, 3, 7] because hypergraphs can be the better mathematical models in many practical cases and higher order structures than graphs. On the other hand, tensor is well known as an important tool in applied mathematics and virtually every discipline in the engineering and physical sciences that makes some use of it. So it is a natural thought to study properties of hypergraphs by using the tool of tensor. In 2005, the definition of eigenvalue of a tensor was independently proposed by Lim [16] and Qi [23]. At the same time, several kinds of eigenvalues for tensors had been proposed, such as H-eigenvalues, Z-eigenvalues, E-eigenvalues and N-eigenvalues. In 2007, by Lim [17] the study of hypergraph via its adjacency tensor and its eigenvalues was initiated. Then in 2009, Rota Bulò and Pelillo [26–28] gave new bounds on the clique number of a uniform hypergraph based on analysis of the largest eigenvalue of the adjacency tensor. As we know, the problem to find the clique number of a 2-uniform hypergraph(i.e. graph) is the NP-complete problem [8], and turns out to be even intractable to a k-uniform hypergraph for $k \geq 3$. However, we have a good algorithm for calculating the largest H-eigenvalues of an irreducible nonnegative tensor [20]. Therefore, it is significant to us depict the bounds on the clique number related to the largest H-eigenvalues for k-uniform hypergraphs.

In this paper, we study some relations between the Laplacian/signless Laplacian H-eigenvalues and the clique/coclique numbers of uniform hypergraphs. The Laplacian/signless Laplacian H-eigenvalues of a uniform hypergraph refer to respectively

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the H-eigenvalues of the Laplacian/signless Laplacian tensors of this uniform hypergraph. This work is motivated by the classic results for graphs [2,9,18,19,21,29], the results of Rota Bulò and Pelillo [28] and the latest results of Yi and Chang [34]. Recently, several papers appeared on nonnegative tensors and spectral hypergraph theory via tensors [4, 6, 10–17, 22–28, 30–34]. Among them, Cooper et al [6] and Qi [24] respectively systematically studied the adjacency tensors, Laplacian and signless Laplacian tensors of uniform hypergraphs. These three notions of tensors are more natural and simpler than those in the literature, so we follow these three notions of tensors throughout the sequel discussion.

The rest of this paper is organized as follows. In the next section, we restatement some definitions on eigenvalues of tensors and uniform hypergraphs. Also we give the definitions and some known results on clique and coclique numbers of a uniform hypergraph. We discuss in Section 3 some inequality relations between the Laplacian/signless Laplacian H-eigenvalues and the clique number of a uniform hypergraph. In Section 4, we present some inequality relations between the Laplacian/ signless Laplacian H-eigenvalues and the coclique number of a uniform hypergraph. Also we give some bounds on the sum of the largest/smallest adjacency/Laplacian/signless Laplacian H-eigenvalues of a hypergraph and its complement hypergraph.

2. Preliminaries

Some definitions of eigenvalues of tensors and uniform hypergraphs are presented in this section.

2.1. H-Eigenvalues of tensors. In this subsection, some basic definitions on H-eigenvalues of tensors are reviewed. For comprehensive references, see [10, 23] and references therein. Especially, for spectral hypergraph theory oriented facts on H-eigenvalues of tensors, please see [12, 24].

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n$ the n-dimensional real space. $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. $\mathbb{R}^n_{++}$ denotes the positive orthant of $\mathbb{R}^n$. For integers $k \geq 3$ and $n \geq 2$, a real tensor $\mathcal{T} = (t_{i_1 \ldots i_k})$ of order k and dimension n refers to a multiway array (also called hypermatrix) with entries $t_{i_1 \ldots i_k}$ such that $t_{i_1 \ldots i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1, \ldots, n\}$ and $j \in [k]$. Tensors are always referred to k-th order real tensors in this paper, and the dimensions will be clear from the content. Given a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{T}\mathbf{x}^k$ is defined as $\sum\limits_{i_1, i_2, \ldots, i_k \in [n]} t_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ and $\mathcal{T}\mathbf{x}^{k-1}$ is defined as an n-dimensional vector such that its i-th element being $\sum\limits_{i_2, \ldots, i_k \in [n]} t_{i i_2 \ldots i_k} x_{i_2} \cdots x_{i_k}$ for all $i \in [n]$. Let $\mathcal{I}$ be the identity tensor of appropriate dimension, e.g., $i_{i_1 \ldots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise when the dimension is n. The following definition was introduced by Qi [23].

Definition 2.1. *Let $\mathcal{T}$ be a k-th order n-dimensional real tensor. For some $\lambda \in \mathbb{R}$, if eigenvalue equation $(\lambda\mathcal{I} - \mathcal{T})\mathbf{x}^{k-1} = 0$ has a solution $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, then λ is called an H-eigenvalue and $\mathbf{x}$ an H-eigenvector associated to $\lambda = \frac{\mathcal{T}\mathbf{x}^k}{||\mathbf{x}||_k^k}$. Furthermore, if $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, then we say that λ is an H^+-eigenvalue of $\mathcal{T}$.*

It is seen that H-eigenvalues are real numbers [23]. By [10, 23], we have that the number of H-eigenvalues of a real tensor is finite. By [24], we have that all the tensors considered in this paper have at least one H-eigenvalue. Hence, we can denote by $\lambda(\mathcal{T})$ (respectively $\mu(\mathcal{T})$) as the largest (respectively smallest) H-eigenvalue of a real tensor $\mathcal{T}$.