INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 12, Number 2, Pages 343–365

SCHEMES AND ESTIMATES FOR THE LONG-TIME NUMERICAL SOLUTION OF MAXWELL'S EQUATIONS FOR LORENTZ METAMATERIALS

JICHUN LI AND SIMON SHAW

Abstract. We consider time domain formulations of Maxwell's equations for the Lorentz model for metamaterials. The field equations are considered in two different forms which have either six or four unknown vector fields. In each case we use arguments tuned to the physical laws to derive data-stability estimates which do not require Gronwall's inequality. The resulting estimates are, in this sense, sharp. We also give fully discrete formulations for each case and extend the sharp data-stability to these. Since the physical problem is linear it follows (and we show this with examples) that this stability property is also reflected in the constants appearing in the *a priori* error bounds. By removing the exponential growth in time from these estimates we conclude that these schemes can be used with confidence for the long-time numerical simulation of Lorentz metamaterials.

Key words. Maxwell's equations, Lorentz model, metamaterial, Galerkin and mixed finite element method, long-time integration, time stepping.

1. Introduction

Electromagnetic metamaterials are artificially structured materials which exhibit exotic properties such as negative refractive index and reversed Doppler effects. The successful construction of such metamaterials in 2000 triggered a wave of further study of metamaterials and exploration of their applications in diverse areas such as sub-wavelength imaging and cloaking. More details can be found in monographs such as [9, 28, 34, 7] and references cited therein.

Although the finite element approximation of Maxwell's equations has been extensively documented for 'classical' materials (see, for example, [3, 4, 8, 14, 31, 33, 37] and their references), there is now an opportunity to build on this body of knowledge for the development and analysis of finite element methods (FEM) for Maxwell's equations for metamaterials. In this direction we mention [10, 11, 5, 2, 21] for the time-harmonic form, and [19, 20, 16] for the time-domain form. Our focus here is on the Lorentz model which, as we will see below, introduces additional unknowns for electrical and magnetic polarizations. These are governed by ordinary differential equations (in time) which hold at each point in space and have the effect of making the (meta)material dispersive, or 'frequency dependent'. In this context we recall also the work on the time-domain Maxwell's equations in general dispersive media in [1, 17, 24, 35, 27, 36]. In particular, [1] contains a study of numerical dispersion for Debye and Lorentz media and [35] gives long-time stability and error estimates for a Debye model.

Received by the editors February 12, 2014, and in revised form March 22, 2014.
2000 *Mathematics Subject Classification.* 65N30, 35L15, 78-08.
This work was supported in part by NSFC Project 11271310, NSF grant DMS-1416742, and a grant from the Simons Foundation (#281296 to Li), in part by scheme 4 London Mathematical Society funding and in part by the Engineering and Physical Sciences Research Council (EP/H011072/1 to Shaw). This support is gratefully acknowledged. Li wants to thank UNLV for granting his sabbatical leave in Spring 2014 when he got time finishing this work.

In recent years there have been several efforts in developing and analyzing some FEMs for the time-domain Maxwell's equations for Lorentz metamaterials (see, for example, [22] and the references therein). However most of these previous results for data-stability and error bounds were derived with the use of Gronwall-type inequalities and, hence, are of limited practical use due to the exponential growth, in time, of the constants. This article improves upon this current 'state of the art' by building upon the 'long-time' results in [35] for two popular numerical schemes.

To be precise, in Section 2 we describe the time domain formulation of Maxwell's equations for Lorentz metamaterials. In Sections 3 and 4, respectively, the field equations are considered in two different forms which have, respectively, six and four unknown vector fields. In each case we use arguments tuned to the physical laws to derive data-stability estimates which do not require Gronwall's inequality. The resulting estimates are sharp, in that they contain stability constants that are time independent, and appear to be novel. We also give fully discrete formulations for each case and extend the sharp data stability to these formulations. Moreover, since the physical problem is linear the error terms obey essentially the same stability estimates but with data replaced by approximation error. With this in mind we can therefore show by examples that the long-time stability properties of these schemes are also reflected in the *a priori* error bounds. The time dependence in these constants then arises from the time dependence in the norms of the data and exact solution and produces, at worst, low-order-polynomial growth in time rather than the exponential growth that arises from Gronwall arguments. Hence, we can conclude that the resulting numerical schemes can be used with confidence for the long time numerical simulation of Lorentz metamaterials. This is the major contribution of the work presented below. In Section 5 we close with a short discussion of the formulations.

Throughout our notation is mostly standard. For example, $C \geqslant 0$ will denote a generic positive constant (independent of the finite element mesh size h and time step size τ) and we let $(H^\sigma(\Omega))^3$ be the standard Sobolev space equipped with the norm $\|\cdot\|_\sigma$ and semi-norm $|\cdot|_\sigma$. Specifically, $\|\cdot\|_0$ will mean the $(L^2(\Omega))^3$-norm. From [31] (for example) we also recall the standard spaces for Maxwell problems,

$$\begin{aligned} H(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in (L^2(\Omega))^3 : \nabla \times \boldsymbol{v} \in (L^2(\Omega))^3\}, \\ H_0(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in H(\mathrm{curl};\Omega) : \mathbf{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega\}, \\ H^\sigma(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in (H^\sigma(\Omega))^3 : \nabla \times \boldsymbol{v} \in (H^\sigma(\Omega))^3\}, \end{aligned}$$

where $\sigma \geqslant 0$ is a real number, and Ω is a bounded Lipschitz polyhedral domain in $\mathcal{R}^3$ with connected boundary $\partial\Omega$ and outward directed unit normal $\mathbf{n}$. We equip $H(\mathrm{curl};\Omega)$ with norm $\|\boldsymbol{v}\|_{0,\mathrm{curl}} = (\|\boldsymbol{v}\|_0^2 + \|\mathrm{curl}\, \boldsymbol{v}\|_0^2)^{1/2}$, and $H^\sigma(\mathrm{curl};\Omega)$ with norm $\|\boldsymbol{v}\|_{\sigma,\mathrm{curl}} = (\|\boldsymbol{v}\|_\sigma^2 + \|\mathrm{curl}\, \boldsymbol{v}\|_\sigma^2)^{1/2}$. For clarity, in the rest of the paper we introduce the vector notation $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^3$ and $\boldsymbol{H}^\sigma(\Omega) = (H^\sigma(\Omega))^3$ and also we often omit the explicit display of the dependence of quantitites on $\boldsymbol{x} \in \Omega$ because we want to focus on the handling of their time dependence. The spatial dependencies are handled in a standard way. Further notation is introduced as and when needed.

2. The governing equations

In general terms, the problem of electromagnetic wave propagation requires the solution of Maxwell's equations,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \text{and} \quad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \quad \text{in } \Omega \times I \tag{1}$$