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IMPROVED ERROR ESTIMATES OF A FINITE DIFFERENCE/SPECTRAL METHOD FOR TIME-FRACTIONAL DIFFUSION EQUATIONS

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Abstract. In this paper, we first consider the numerical method that Lin and Xu proposed and analyzed in [Finite difference/spectral approximations for the time-fractional diffusion equation, JCP 2007] for the time-fractional diffusion equation. It is a method basing on the combination of a finite different scheme in time and spectral method in space. The numerical analysis carried out in that paper showed that the scheme is of $(2-\alpha)$-order convergence in time and spectral accuracy in space for smooth solutions, where α is the time-fractional derivative order. The main purpose of this paper consists in refining the analysis and providing a sharper estimate for both time and space errors. More precisely, we improve the error estimates by giving a more accurate coefficient in the time error term and removing the factor in the space error term, which grows with decreasing time step. Then the theoretical results are validated by a number of numerical tests.

Key words. Error estimates, finite difference methods, spectral methods, time fractional diffusion equation.

1. Introduction

As a powerful tool in modelling the phenomenon related to nonlocality and spatial heterogeneity, the fractional partial differential equations (FPDE for short hereafter) has been attracting increasing attention in recent years. They are now finding its many applications in a broad range of fields such as control theory, biology, electrochemical processes, viscoelastic materials, polymer, finance, and etc; see, e.g.,[1, 2, 4, 5, 6, 8, 9, 12, 13, 19, 23, 25] and the references therein.

Similar to the role of the heat equation in traditional modelling, the time-fractional diffusion equation considered in this paper is of importance not only in its own right, but also it constitutes the kernel of many other more general FPDE. This model equation governs the evolution for the probability density function that describes anomalously diffusing particles. For some fractional models, we mention, e.g., the chaotic dynamics charge transport problem in amorphous semiconductors [26, 27], the NMR diffusometry in disordered materials [20], the dynamics of a bead in polymer network [3], and the propagation of mechanical diffusive waves in viscoelastic media [18]. For more applications where the time-fractional diffusion appears, we refer to a generalized diffusion equation which describes transport processes with long memory [10]; the physical model of water transport in soil, which is a generalized Richards' equation with time-fractional derivative [21]; the similarity problem of nonlinear integro-differential type [22], etc.

There have been a number of numerical methods constructed for the time-fractional diffusion equations. We mention, among others, the work [17] by Liu

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et al. on the finite difference method in both space and time, a finite difference scheme for the fractional diffusion-wave equation by Sun and Wu [29], a L1 scheme used to approximate the fractional order time derivative by Langlands and Henry [14], a particle tracking approach by Zhang et al. [30], an alternating direction implicit scheme by Zhang and Sun [31], finite difference schemes for a variable-order equation by Sun et al. [28], and convergence analysis of the finite element method in Jin et al. [11].

On one side, fractional derivatives are non-local operators, which explains one of their most significant uses in applications: they possess a memory effect which is present in several materials such as viscoelastic materials or polymers. On the other side, the nonlocality of the fractional derivatives makes the design of accurate and fast methods difficult. In particular, the fact that all previous solutions have to be saved to compute the solution at the current time point would make the storage very expensive if a low-order method is employed. This consideration has inspired some recent work [15, 16] on developing spectral methods for time-fractional differential equations. Particularly, Lin and Xu [16] proposed a finite difference scheme in time and Legendre spectral method in space for the time-fractional diffusion equation. A convergence rate of $(2-\alpha)$-order in time and spectral accuracy in space of the method was proved, where α is the time derivative order.

In this paper, we follow the work in [16] with an attempt to improve the error estimates obtained therein. The main contribution of the paper is as follows: Firstly, a sharper estimate for both time and space errors is derived by using different analysis techniques. Specifically, we obtain a more accurate coefficient in front of the time error term and remove the undesirable factor in the space error term, which grows with decreasing time step. Secondly, this new estimate is confirmed by a number of numerical tests carefully designed for the verification.

The outline of this paper is as follows. In the next section we first describe the time discretization for the time-fractional diffusion equation, then derive the truncation error. In Section 3 we describe two spectral methods for the space discretization, and derive the full discrete error estimates. Some numerical examples are given in Section 4. Finally we give some concluding remarks in Section 5.

2. A $2-\alpha$ order finite difference scheme in time

We first describe the problem of fractional differential equations that is studied in this paper. Let $T > 0$, $\Lambda = (-1, 1), I = (0, T]$, consider the time-fractional diffusion equation of the form

$$\partial_t^\alpha u(x,t) - \partial_x^2 u(x,t) = 0, \quad x \in \Lambda,\ t \in I, \tag{1}$$

subject to the following initial and boundary conditions:

$$u(x,0) = g(x), \quad x \in \Lambda, \tag{2}$$

$$u(-1,t) = u(1,t) = 0, \quad 0 \le t \le T, \tag{3}$$

where α is the order of the time-fractional derivative. Here, we consider the case $0 < \alpha < 1$ and fractional derivative in the Caputo sense [23], defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \partial_s u(x,s) \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1.$$