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FAST SOLVERS FOR THE SYMMETRIC IPDG DISCRETIZATION OF SECOND ORDER ELLIPTIC PROBLEMS

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Abstract. In this paper, we develop and analyze a preconditioning technique and an iterative solver for the linear systems resulting from the discretization of second order elliptic problems by the symmetric interior penalty discontinuous Galerkin methods. The main ingredient of our approach is a stable decomposition of the piecewise polynomial discontinuous finite element space of arbitrary order into a linear conforming space and a space containing high frequency components. To derive such decomposition, we introduce a novel interpolation operator which projects piecewise polynomials of arbitrary order to continuous piecewise linear functions. We prove that this operator is stable which allows us to derive the required space decomposition easily. Moreover, we prove that both the condition number of the preconditioned system and the convergent rate of the iterative method are independent of the mesh size. Numerical experiments are also shown to confirm these theoretical results.

Key words. Discontinuous Galerkin methods, iterative method, preconditioner.

1. Introduction

Discontinuous Galerkin (DG) methods are widely used numerical methodologies for the numerical solutions of partial differential equations. They have traditionally been used for the numerical solutions of hyperbolic equations [36, 29, 23, 24, 19]. There are many advantages in using DG methods compared with other types of finite element methods. For example, DG methods allow more flexibility in handling equations whose types change within the computational domain and in designing *hp*-refinement strategies. Besides, they have the ability to provide important conservation properties as well as give block diagonal mass matrices for time-dependent problems [23, 24, 19]. Owing to these unique advantages, DG methods have also been developed for second order elliptic problems [14, 18, 25] and many other problems. In addition, DG methods based on staggered grids are recently developed and analyzed for a large class of problems [21, 22, 26, 27, 28]. On the other hand, one main obstacle in the efficient implementation of DG methods is that the resulting linear systems contain a larger number of unknowns compared with conforming methods. Thus, the construction of fast algorithms is crucial for the efficient implementations of DG methods. In this paper, we will pay our attention to the symmetric interior penalty discontinuous Galerkin (IPDG) for second order elliptic equations.

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Fast solvers for DG methods have been widely studied in the literature. For example, additive Schwarz methods [31, 1, 2, 6], multigrid methods [13, 11, 10, 5, 9, 4] and preconditioning techniques [32, 30, 15, 16, 5, 4] are developed for the efficient solution strategies for DG methods. The first work on preconditioning technique, to the best of our knowledge, is given by Gopalakrishnan and Kanschat [32], in which they studied the variable V-cycle multigrid operator as a preconditioner, but assumed that the underlying hierarchy of meshes is quasi-uniform and the solution exhibits a certain (weak) extra regularity. Dobrev et. al. presented a two-level scheme in the framework of the auxiliary space method in [30]. However, they only analyzed the case with discontinuous piecewise linear finite elements and their technique requires the exact solution of a coarse-grid problem. Brix et. al. constructed a multilevel preconditioner and obtained uniformly bounded condition numbers for the preconditioned linear system without the use of the weaker assumptions presented in [15, 16]. Their scheme allows the use of triangulations with hanging nodes or graded meshes. Recently, some iterative and preconditioning techniques are presented and analyzed in [5]. The key idea is a splitting of the discontinuous finite element space into the standard Crouziex-Raviart space and its complementary space with respect to the energy inner product induced by IPDG-0 methods. Such decomposition has also been proposed and used in [17] for obtaining a priori error bounds for some DG methods. Moreover, the results in [5] are extended in [4] to the design of multilevel preconditioners for linear systems resulting from the DG discretization of elliptic problems with discontinuous coefficients. However, the mathematical analysis of these methods is based on the discretization by using discontinuous piecewise linear finite element spaces.

In this paper, we will develop and analyze a preconditioning technique and an iterative method for solving the linear systems resulting from the discretization of elliptic boundary value problems by symmetric IPDG methods. The key to the constructions of these is a stable space decomposition of the discontinuous finite element space $\mathbb{V}_h$ containing piecewise polynomials. More precisely, we will prove the following stable splitting

$$\mathbb{V}_h = \sum_{i=1}^{N} \mathbb{V}_i + \mathbb{V}_h^{Conf},$$

where $\mathbb{V}_i = \text{span}\{\varphi_i\}$, $\{\varphi_i\}_{i=1}^N$ is the set of all nodal basis functions in $\mathbb{V}_h$ having dimension N and $\mathbb{V}_h^{Conf}$ denotes the conforming finite element space with homogeneous Dirichlet boundary conditions. The above decomposition can be seen as decomposing the finite element space $\mathbb{V}_h$ as the sum of conforming space, whose fast solution techniques are well-known, and the space $\sum_{i=1}^N \mathbb{V}_i$, which can be regarded as a space containing high frequency components. We will prove that these high frequency components can be handled by using Jacobi or Gauss-Seidel smoothers. The use of this type of space decomposition can also be found in [30, 15, 16, 4]. In this paper, we will introduce a new interpolation operator, which gives a simpler approach for establishing the aforementioned stable space decomposition. We will show that the condition number of the preconditioned linear system is uniformly bounded and the iterative method is uniformly convergent with respect to the mesh size. Compared with most of the existing works, our main contributions in this paper are threefold. First, we give a construction of an interpolation operator of Scott-Zhang type, which from $\mathbb{V}_h$ into $\mathbb{V}_h^{Conf}$. Secondly, our ideas can be applied directly to the original discrete variational problems without the need of another equivalent bilinear form. Last and more importantly, our preconditioning technique