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A POSTERIORI ERROR ESTIMATES OF FINITE VOLUME ELEMENT METHOD FOR SECOND-ORDER QUASILINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper, we consider the a posteriori error estimates of the finite volume element method for the general second-order quasilinear elliptic problems over a convex polygonal domain in the plane, propose a residual-based error estimator and derive the global upper and local lower bounds on the approximation error in the H^1-norm. Moreover, for some special quasilinear elliptic problems, we propose a residual-based a posteriori error estimator and derive the global upper bound on the error in the L^2-norm. Numerical experiments are also provided to verify our theoretical results.

Key words. quasilinear elliptic problem, finite volume element method, a posteriori error estimates.

1. Introduction

The finite volume element method (FVEM, also called finite volume method or covolume method in some literature) is a class of important numerical tools for solving differential equations, especially for those arising from physical conservation laws including mass, momentum, and energy. Because this method possesses local physical conservation property, which is crucial in many applications, it is popular in computational fluid mechanics. In the past several decades, many researchers have studied this method extensively and obtained some important results. We refer to monograph [30] for the general presentation of this method, and to [3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 22, 23, 26, 27, 28, 29, 31, 32, 35, 36, 38, 39, 40, 41, 42] and references therein for details.

In this paper, we study the a posteriori error estimates of the finite volume element method for the second-order quasilinear elliptic boundary value problems

$$(1) \qquad \begin{cases} \mathcal{L}u = -\nabla\cdot F(x,\nabla u) + g(x,u,\nabla u) = 0, & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We assume that $F(x,z) : \overline{\Omega}\times\mathbb{R}^2 \to \mathbb{R}^2$ and $g(x,y,z) : \overline{\Omega}\times\mathbb{R}^1\times\mathbb{R}^2 \to \mathbb{R}^1$ are smooth functions and that (1) has a solution $u \in H_0^1(\Omega)\cap W^{2,r}(\Omega)$ for some $r > 2$. The smoothness requirements on those functions will be given in detail later.

There are some important numerical results available for (1). We refer the reader to [20, 33, 37] for the finite element method and to [25] for the hp-discontinuous Galerkin methods.

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Until now, the finite volume element method for the linear elliptic problems has been well understood. However, there are very few works related to the nonlinear elliptic problems. To the best of our knowledge, the authors of [5, 6, 14, 29] studied the finite volume element method and developed some a priori error estimates only for the following quasilinear elliptic problems

$$-\nabla \cdot (\lambda(u)\nabla u) = f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega, \tag{2}$$

where λ is a smooth scalar function. Recently, Bi and Ginting [8] considered the finite volume element method for (1), proved the existence and uniqueness of the finite volume element solutions under the assumption $u \in W^{2,r}(\Omega), r > 2$, where r may be close to 2, and derived the a priori error estimates in the H^1-, L^2-, $W^{1,\infty}$- and L^∞-norms.

Compared with the relatively mature a posteriori error estimates of the finite element method, the a posteriori error analysis of the finite volume element method is still under development, and until now only a few results have been obtained. We mention [12, 28] for the linear elliptic problems. However, for the nonlinear elliptic problems, there are only [4] and [7] available. The authors of [4, 7] established the residual-based a posteriori error estimates of the finite volume element method, respectively, for (2) and

$$-\nabla \cdot (A(u)\nabla u) = f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega, \tag{3}$$

where $A(u)$ is a smooth and bounded uniformly positive definite matrix.

As a subsequent work of [8], in this paper, we study the a posteriori error estimates of the finite volume element method for (1) and propose a natural and computationally easy residual-based H^1-norm a posteriori error estimator. Under two assumptions that $u \in W^{2,r}(\Omega), r > 2$, and the mesh parameter is sufficiently small, we derive the global upper bound and local lower bounds on the error. Moreover, for some special problems (1) which satisfy $D_{zz}F = 0$ and $D_{zz}g = 0$, we propose a residual-based L^2-norm a posteriori error estimator and derive the global upper bound on the error. We point out that the two assumptions above are reasonable, which guarantee the existence of the finite volume element approximations of (1), see [8] for details.

In the present work, for the sake of simplicity, we focus our attention on the quasilinear problems on a polygonal domain, which is the same as those in [20, 33, 37]. Smooth boundaries are important for many nonlinear problems as even theoretical results are not always available on polygonal domains. However, the proper treatment of the curved boundary is somewhat technical (see [21] for details) and we don't wish to clutter our presentation.

The organization of this paper is as follows. In section 2, we introduce some notation, formulate the finite volume element method for (1), and give some lemmas used in the subsequent analysis. In section 3, we propose a residual-based H^1-norm a posteriori error estimator of the finite volume element method for (1) and derive the global upper bound and local lower bounds on the error. In section 4, for some special problems (1) which satisfy $D_{zz}F = 0$ and $D_{zz}g = 0$, such as Bratu's equation and some nonlinear eigenvalue equations, we propose a residual-based L^2-norm a posteriori error estimator and derive the global upper bound on the error. In section 5, we provide two numerical experiments that confirm our theoretical findings in this paper. Finally, in Section 6, we summarize the main results of this paper and draw some conclusions.