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NUMERICAL ANALYSIS AND TESTING OF A FULLY DISCRETE, DECOUPLED PENALTY-PROJECTION ALGORITHM FOR MHD IN ELSÄSSER VARIABLE

MINE AKBAS, SONGUL KAYA, MUHAMMAD MOHEBUJJAMAN, AND LEO G. REBHOLZ

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Abstract. We consider a fully discrete, efficient algorithm for magnetohydrodynamic (MHD) flow that is based on the Elsässer variable formulation and a timestepping scheme that decouples the MHD system but still provides unconditional stability with respect to the timestep. We prove stability and optimal convergence of the scheme, and also connect the scheme to one based on handling each decoupled system with a penalty-projection method. Numerical experiments are given which verify all predicted convergence rates of our analysis on some analytical test problems, show the results of the scheme on a set of channel flow problems match well the results found when the computation is done with MHD in primitive variable, and finally show the scheme performs well on a channel flow over a step.

Key words. Magnetohydrodynamics, Elsässer variables, Penalty-projection method, finite element method

1. Introduction

We consider the efficient and accurate numerical approximation of magnetohydrodynamic (MHD) flow, which is governed by the system of evolution equations [19, 5]

$$u_t + (u\cdot\nabla)u - s(B\cdot\nabla)B - \nu\Delta u + \nabla p = f, \tag{1}$$

$$\nabla\cdot u = 0, \tag{2}$$

$$B_t + (u\cdot\nabla)B - (B\cdot\nabla)u - \nu_m\Delta B + \nabla\lambda = \nabla\times g, \tag{3}$$

$$\nabla\cdot B = 0, \tag{4}$$

in $\Omega\times(0,T)$, where Ω is the domain of the fluid, u is the velocity of the fluid, p is a modified pressure, B is the magnetic field, s is the coupling number, ν is the kinematic viscosity, ν_m is the magnetic resistivity, f is the body force, and $\nabla\times g$ is the forcing on the magnetic field. The physical principles governing such flows are that when an electrically conducting fluid moves in a magnetic field, the magnetic field induces currents in the fluid, which in turn creates forces on the fluid and also alters the magnetic field. In the recent years, the study of MHD flows has become important due to applications in, e.g. astrophysics and geophysics [17, 23, 12, 10, 4, 6], liquid metal cooling of nuclear reactors [3, 15, 26], and process metallurgy [8].

A fundamental difficulty in simulating MHD flow is solving the fully coupled linear systems that arise in common discretizations of (1)-(4). It is an open problem how to decouple the equations in an unconditionally stable way (with respect to the timestep size), and thus timestepping methods that decouple the equations are prone to unstable behavior without using excessively small timestep sizes. To

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confront this issue, an excellent idea was presented by Trenchea in [27]: if one rewrites the MHD system in terms of Elsässer variables (defined below), then an unconditionally stable, decoupled, timestepping algorithm can be created. Analysis of this algorithm in a semidiscrete setting (temporal discretization only) with a defect correction method was performed in [28], but no numerical experiments were performed beyond convergence rate verification. The purpose of this paper is 1) to analyze and test Trenchea's algorithm in a fully discrete setting, i.e. together with a finite element spatial discretization, 2) to extend the algorithm and analysis to a more efficient class of timestepping algorithms (penalty-projection type), and 3) test the algorithms on some benchmark problems and compare to simulations with primitive variables.

The Elsässer formulation of MHD was first proposed by W. Elsässer in 1950 [11], and since then has been used in several analytical studies, e.g. [25, 9, 22]. To derive it, begin by splitting the magnetic field into two parts, $\sqrt{s}B =: \sqrt{s}B_0 + \sqrt{s}b$ (mean and fluctuation, respectively), with $B_0 = B_0(t)$. For boundary conditions, we assume the Dirichlet condition $B = B_0$ on $\partial\Omega$, and homogeneous Dirichlet conditions for the velocity, $u = 0$, and magnetic field fluctuations, $b = 0$. The system (1)-(4) can now be written as

$$u_t + (u\cdot\nabla)u - s(B_0\cdot\nabla)b - s(b\cdot\nabla)b - \nu\Delta u + \nabla p = f, \tag{5}$$

$$\nabla\cdot u = 0, \tag{6}$$

$$b_t + (u\cdot\nabla)b - (B_0\cdot\nabla)u - (b\cdot\nabla)u - \nu_m\Delta b + \nabla\lambda = \nabla\times g - \frac{dB_0}{dt}, \tag{7}$$

$$\nabla\cdot b = 0. \tag{8}$$

Rescaling (7) by $\sqrt{s}$, adding (subtracting) (5) to (from) (7) and setting $f_1 := f + \nabla\times g - \frac{dB_0}{dt}$, $f_2 := f - \sqrt{s}(\nabla\times g + \frac{dB_0}{dt})$, $q := p + \sqrt{s}\lambda$ and $r := p - \sqrt{s}\lambda$ gives

$$\begin{aligned}
(u+\sqrt{s}b)_t + (u\cdot\nabla)(u+\sqrt{s}b) - (\sqrt{s}B_0\cdot\nabla)(u+\sqrt{s}b)&\\
-(\sqrt{s}b\cdot\nabla)(u+\sqrt{s}b) - \nu\Delta u - \nu_m\Delta(\sqrt{s}b) + \nabla q &= f_1,\\
\nabla\cdot(u+\sqrt{s}b) &= 0,\\
(u-\sqrt{s}b)_t + (u\cdot\nabla)(u-\sqrt{s}b) + (\sqrt{s}B_0\cdot\nabla)(u-\sqrt{s}b)&\\
+(\sqrt{s}b\cdot\nabla)(u-\sqrt{s}b) - \nu\Delta u + \nu_m\Delta(\sqrt{s}b) + \nabla r &= f_2,\\
\nabla\cdot(u-\sqrt{s}b) &= 0.
\end{aligned}$$

Now defining $v = u + \sqrt{s}b$, $w = u - \sqrt{s}b$, $\tilde{B}_0 = \sqrt{s}B_0$ produces the Elsässer formulation

$$v_t + w\cdot\nabla v - (\tilde{B}_0\cdot\nabla)v + \nabla q - \frac{\nu+\nu_m}{2}\Delta v - \frac{\nu-\nu_m}{2}\Delta w = f_1, \tag{9}$$

$$\nabla\cdot v = 0, \tag{10}$$

$$w_t + v\cdot\nabla w + (\tilde{B}_0\cdot\nabla)w + \nabla r - \frac{\nu+\nu_m}{2}\Delta w - \frac{\nu-\nu_m}{2}\Delta v = f_2, \tag{11}$$

$$\nabla\cdot w = 0. \tag{12}$$

This paper is arranged as follows. In section 2, we provide notation and mathematical preliminaries that will allow for a smooth analysis to follow. Section 3 presents the fully discrete scheme, and proves stability and convergence for it. Section 4 presents a penalty-projection variation of the scheme, and proves it is equivalent to the scheme of Section 3 when the the penalty parameter is large. Section 5 presents numerical experiments and conclusions are drawn in section 6.