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NUMERICAL SHOOTING METHODS FOR OPTIMAL BOUNDARY CONTROL AND EXACT BOUNDARY CONTROL OF 1-D WAVE EQUATIONS

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Abstract. Numerical solutions of optimal Dirichlet boundary control problems for linear and semilinear wave equations are studied. The optimal control problem is reformulated as a system of equations (an optimality system) that consists of an initial value problem for the underlying (linear or semilinear) wave equation and a terminal value problem for the adjoint wave equation. The discretized optimality system is solved by a shooting method. The convergence properties of the numerical shooting method in the context of exact controllability are illustrated through computational experiments. In particular, in the case of the linear wave equation, convergent approximations are obtained for both smooth minimum L^2-norm Dirichlet control and generic, non-smooth minimum L^2-norm Dirichlet controls.

Key words. Controllability, optimal control, wave equation, shooting method, finite difference method.

1. Introduction

In this chapter we consider an optimal boundary control approach for solving the exact boundary control problem for one-dimensional linear or semilinear wave equations defined on a time interval $(0,T)$ and spatial interval $(0,X)$. The exact boundary control problem we consider is to seek a boundary control $\mathbf{g} = (g_L, g_R) \in \mathbf{L^2(0,T)} \subset [L^2(0,T)]^2$ and a corresponding state u such that the following system of equations hold:

$$
(1) \qquad \begin{cases} u_{tt} - u_{xx} + f(u) = V \quad \text{in } Q \equiv (0,T) \times (0,X)\,, \\ u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1 \quad \text{in } (0,X)\,, \\ u|_{t=T} = W \quad \text{and} \quad u_t|_{t=T} = Z \quad \text{in } (0,X)\,, \\ u|_{x=0} = g_L \text{ and } u|_{x=1} = g_R \quad \text{in } (0,T)\,, \end{cases}
$$

where u_0 and u_1 are given initial conditions defined on $(0,X)$, $W \in L^2(0,X)$ and $Z \in H^{-1}(0,X)$ are prescribed terminal conditions, V is a given function defined on $(0,T) \times (0,X)$, f is a given function defined on $\mathbb{R}$, and $\mathbf{g} = (g_L, g_R) \in [L^2(0,T)]^2$ is the boundary control.

It is well known (see, e.g., [15, 16, 18, 19]) that when $f = 0$ (i.e., the equation is linear) and T is sufficiently large, the exact controllability problem (1) admits at least one state-control solution pair $(u, \mathbf{g})$; furthermore, the exact controller $\mathbf{g}$

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having minimum boundary L^2 norm is unique. Exact boundary controllability for semilinear wave equations have also been established for certain asymptotically linear or superlinear f; see, e.g., [4, 23, 24].

For the exact boundary controllability problem associated with the linear wave equation there are basically two classes of computational methods in the literature. The first class is HUM-based methods; see, e.g., [6, 9, 15, 17, 22]. The approximate solutions obtained by the HUM-based methods in general do not seem to converge (even in a weak sense) to the exact solutions as the temporal and spatial grid sizes tend to zero. Methods of regularization including Tychonoff regularization and filtering that result in convergent approximations were introduced in those papers on HUM-based methods. The second class of computational methods for boundary controllability of the linear wave equation was those based on the method proposed in [8]. One solves a discrete optimization problem that involves the minimization of the discrete boundary L^2 norm subject to the undetermined linear system of equations formed by the discretization of the wave equation and the initial and terminal conditions. This approach was implemented in [12]. The computational results demonstrated the convergence of the discrete solutions when the exact minimum boundary L^2 norm solution is smooth. In the generic case of a non-smooth exact minimum boundary L^2 norm solution the computational results of [12] exhibited at least a weak L^2 convergence of the discrete solutions.

Although there are well-known theoretical results concerning boundary controllability of *semilinear* wave equations (see, e.g., [4, 23, 24]), little seems to exist in the literature about computational methods for such problems.

In this chapter we attempt to solve the exact controllability problems by an optimal control approach. Precisely, we consider the following optimal control problem: minimize the cost functional

$$\begin{aligned}\mathcal{J}_0(u,\mathbf{g}) =& \frac{\sigma}{2}\int_0^1 |u(T,x)-W(x)|^2\,dx + \frac{\tau}{2}\int_0^1 |u_t(T,x)-Z(x)|^2\,dx \\ &+ \frac{1}{2}\int_0^1 (|g_L|^2+|g_R|^2)\,dt\end{aligned} \tag{2}$$

subject to

$$\begin{cases} u_{tt}-u_{xx}+f(u)=V \quad \text{in } Q\equiv(0,T)\times(0,1) \\ u|_{t=0}=u_0 \quad \text{and} \quad u_t|_{t=0}=u_1 \quad \text{in } (0,1) \\ u|_{x=0}=g_L \text{ and } u|_{x=1}=g_R \quad \text{in } (0,T)\,. \end{cases} \tag{3}$$

The optimal control problem is converted into an optimality system of equations and this optimality system of equations will be solved by a shooting method.

The optimal control approach of this chapter provides an alternative method to the two classes of methods mentioned in the foregoing for solving the exact controllability problem for the linear wave equations; it also offers a systematic procedure for solving exact controllability problems for the semilinear wave equations. The computational solutions of this chapter obtained by an optimal control approach exhibit behaviors similar to those of the solutions obtained in [12]. Note that an optimal solution exists even when the equation is not exactly controllable. Note also that the solution methods in the literature for optimal control of PDEs can be utilized. and that there are certain intrinsic parallelisms to the algorithms studied in this chapter.