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FINITE ELEMENT METHOD AND ITS ERROR ESTIMATES FOR THE TIME OPTIMAL CONTROLS OF HEAT EQUATION

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Abstract. In this paper, we discuss the time optimal control problems governed by heat equation. The variational discretization concept is introduced for the approximation of the control, and the semi-discrete finite element method is applied for the controlled heat equation. We prove optimal a priori error estimate for the optimal time T, and quasi-optimal estimates for the optimal control u, the related state y and adjoint state p.

Key words. Time optimal control problems, finite element method, error estimates.

1. Introduction

One of the most important optimal control problems is the optimal time control problem. There have been extensive researches on the theoretical parts of the time optimal control problems of ODEs (see, e.g., [4] and [5]) and time-dependent PDEs (see [14, 15, 17] and the references cited therein), but only a few works related to their numerical algorithms can be found, especially the finite element approximations and error estimates for PDEs, among them we should mention the work [8], [9] and [16].

The purpose of this work is to investigate the finite element approximations of the time optimal control problem governed by heat equation. The model problem that we shall investigate is the following time optimal control problem:

$$\min_{u\in U_{ad}}\Big\{T: \quad y(T;y_0,u)\in B(0,1)\Big\}, \tag{1}$$

where u is the control, the state y satisfies the following controlled equation:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t}-\Delta y(x,t)=Eu(x,t) & \text{in } \Omega\times(0,+\infty),\\ y(x,t)=0 & \text{on } \partial\Omega\times(0,+\infty),\\ y(x,0)=y_0(x) & \text{in } \Omega. \end{cases} \tag{2}$$

The details will be specified in the next section.

Although the finite element approximations of PDE-constrained optimal control problems and related error estimates are well studied in the decades and there are huge literatures in this aspect (see, e.g., [3], [7], [10], [11], [12] and the references cited therein), the finite element method and its error estimates for time optimal control problems are addressed only in a few papers. To the best of our knowledge, the earliest work can be traced back to [8] and [9]. In both works, the finite element approximations are introduced for the time optimal control problems and the convergence analyses are provided. In [8] Knowles considered the finite dimensional control ($u=\sum_{i=1}^m f_i(t)g_i(x)$, where $g_i(x)$, $i=1,\cdots,m$, are given functions) which acts as the Robin boundary condition of the controlled equation and gave

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the error estimate for the optimal time with order $O(h^{\frac{3}{2}-\delta})$ for an arbitrary small $\delta > 0$. While in [9] Lasiecka proved the convergence (without orders) of the optimal time for the Dirichlet boundary control problem. In a recent work [16] by Wang and Zheng, the time optimal control problem (1)-(2) is discussed. The error estimate for the optimal time with order $O(h+l)$ is provided under some additional assumptions. These assumptions are not easy to verify in general cases. In this paper, we introduce the variational discretization concept (see, e.g., [6] and [7]) for the approximation of the control. Using this scheme, the error analysis becomes easier and the optimal error estimate for the optimal time is proved without those complicated assumptions. Moreover, the error estimates for the optimal control and the state are also provided in this paper which are not found elsewhere.

The plan of the paper is as follows. In section 2, we introduce the model time optimal control problem and construct its finite element approximation. The error estimates for the time optimal control problems are then analyzed in section 3, where the error estimates for the optimal time T, the optimal control u, the related state y and adjoint state p are provided. Finally, we give a conclusion to the results obtained in this paper and an outlook for some possible further works in the last section.

2. Time optimal control problem and its finite element approximation

In this section, we formulate the model optimal control problem and its finite element approximation.

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a convex and bounded domain with sufficiently smooth boundary $\partial\Omega$. In the rest of the paper, we shall take the control space $U = L^\infty(0,+\infty; L^2(\omega))$ with $\bar{\omega} \subset \Omega$. We use the standard norms $\|\cdot\|_{C([a,b];L^2(\Omega))}$ and $\|\cdot\|_{L^2(a,b;L^2(\Omega))}$ for related Sobolev spaces. For simplicity, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the usual norm and the inner product of $L^2(\Omega)$, respectively. In addition, C denotes a general positive constant independent of the mesh size h.

Let

$$U_{ad} = \{v \in L^\infty(0,+\infty; L^2(\omega)) : \quad \|v(t)\|_{L^2(\omega)} \le 1 \text{ for almost every } t \in [0,+\infty)\},$$

$$B(0,1) = \{w \in L^2(\Omega) : \quad \|w\| \le 1\}.$$

Then the model problem that we shall investigate is the following time optimal control problem (see [16]):

$$\min_{u \in U_{ad}} \Big\{T : \quad y(T; y_0, u) \in B(0,1)\Big\} \tag{3}$$

with $y(\cdot; y_0, u)$ the unique solution of the following equation

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} - \Delta y(x,t) = Eu(x,t) & \text{in } \Omega \times (0,+\infty), \\ y(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{4}$$

corresponding to the control u and the initial value y_0, here

$$E(f)(x) = \begin{cases} f(x) & \text{if} \quad x \in \omega, \\ 0 & \text{if} \quad x \in \Omega \setminus \omega. \end{cases}$$

Throughout this paper, we will treat the solutions of (4) as functions of the time variable t, from $\mathbb{R}^+ := [0,+\infty)$ to the state space $L^2(\Omega)$. We call the number $\tilde{T}(y_0) := \min_{u \in U_{ad}} \{T : y(T; y_0, u) \in B(0,1)\}$ the optimal time, while a control $\tilde{u} \in U_{ad}$, and satisfying the property that $y(\tilde{T}(y_0); y_0, \tilde{u}) \in B(0,1)$, is called an optimal control with corresponding optimal state $\tilde{y} := y(\cdot; y_0, \tilde{u})$. Clearly, $\tilde{T}(\cdot)$ defines a