INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 2, Pages 296–309

NUMERICAL SOLUTION OF NONSTATIONARY PROBLEMS FOR A CONVECTION AND A SPACE-FRACTIONAL DIFFUSION EQUATION

PETR VABISHCHEVICH

Abstract. Convection-diffusion equations provide the basis for describing heat and mass transfer phenomena as well as processes of continuum mechanics. An unsteady problem is considered for a convection and a space-fractional diffusion equation in a bounded domain. A first-order evolutionary equation containing a fractional power of an elliptic operator of second order is studied for general boundary conditions of Robin type. Finite element approximation in space is employed. To construct approximation in time, regularized two-level schemes are used. The numerical implementation is based on solving the equation with the fractional power of the elliptic operator using an auxiliary Cauchy problem for a pseudo-parabolic equation. The results of numerical experiments are presented for a model two-dimensional problem.

Key words. Convection-diffusion problem, fractional partial differential equations, elliptic operator, fractional power of an operator, two-level difference scheme.

1. Introduction

Convection-diffusion problems are typical for mathematical models of fluid mechanics. Heat transfer as well as impurities spreading are occurred not only due to diffusion, but result also from medium motion. Principal features of physical and chemical phenomena observing in fluids and gases [2, 20] are generated by media motion resulting from various forces. Computational algorithms for the numerical solution of such problems are of great importance; they are discussed in many publications (see, e.g., [13, 25]).

In considering the second-order parabolic equations, the convective terms may be written in divergent, nondivergent, and skew-symmetric forms [29]. Transient problems of convection-diffusion are governed by evolutionary operator equations in the appropriate spaces. Their study is based on examining the corresponding properties of the differential operators of convective and diffusive transport. In constructing their discrete analogs, we focus on the approximations that inherit the basic properties of these operators.

Nowadays, non-local applied mathematical models based on the use of fractional derivatives in time and space are actively discussed [1, 8, 18]. Many models, which are used in applied physics, biology, hydrology and finance, involve both sub-diffusion (fractional in time) and supper-diffusion (fractional in space) operators. Supper-diffusion problems are treated as evolutionary problems with a fractional power of an elliptic operator. The mathematical models of convection with fractional diffusion are considered with many works (see., e.g., [3, 11, 21–23, 36]).

To solve problems with fractional powers of elliptic operators, we can apply finite volume and finite element methods oriented to using arbitrary domains and irregular computational grids [19, 26]. The computational realization is associated with

Received by the editors February 9, 2015 and, in revised form, March 15, 2015.

2000 *Mathematics Subject Classification.* 26A33, 35R11, 65F60, 65M06.

This work was supported by the Russian Foundation for Basic Research (project 14-01-00785).

the implementation of the matrix function-vector multiplication. For such problems different approaches [10] are available. Problems of using Krylov subspace methods with the Lanczos approximation when solving systems of linear equations associated with the fractional elliptic equations are discussed in [16]. A comparative analysis of the contour integral method, the extended Krylov subspace method, and the preassigned poles and interpolation nodes method for solving space-fractional reaction-diffusion equations is presented in [6]. The simplest variant is associated with the explicit construction of the solution using the known eigenvalues and eigenfunctions of the elliptic operator with diagonalization of the corresponding matrix [5, 14, 15]. Unfortunately, all these approaches demonstrates too high computational complexity for multidimensional problems.

We have proposed [34] a computational algorithm for solving an equation for fractional powers of elliptic operators on the basis of a transition to a pseudo-parabolic equation. For the auxiliary Cauchy problem, the standard two-level schemes are applied. The computational algorithm is simple for practical use, robust, and applicable to solving a wide class of problems. A small number of time steps is required to find a solution. This computational algorithm for solving equations with fractional powers of operators is promising when considering transient problems.

To solve numerically evolutionary equations of first order, as a rule, two-level difference schemes are used for approximation in time. Investigation of stability for such schemes in the corresponding finite-dimensional (after discretization in space) spaces is based on the general theory of operator-difference schemes [27, 28]. In particular, the backward Euler scheme and Crank-Nicolson scheme are unconditionally stable for a non-negative operator. As for one-dimensional problems for the space-fractional diffusion equation, an analysis of stability and convergence for this equation was conducted in [17] using finite element approximation in space. A similar study for the Crank-Nicolson scheme was considered earlier in [31] using finite difference approximations in space. We separately note the works [12, 24, 30], where the numerical methods for solving one-dimensional in spaces nonstationary problems for a convection and a space-fractional diffusion equation are considered.

In this paper, we propose two-level difference schemes for numerical solution of multidimensional nonstationary problems for a convection and a space-fractional diffusion equation. The main computational complexity of such problems is associated with solving a fractional diffusion problem on the new time level. An algorithm that uses an auxiliary problem for a pseudo-parabolic equation is applied [34]. The stability of the proposed two-level difference schemes is investigated. Results of calculations for a model two-dimensional problem based on finite-element approximations in space demonstrate efficiency of the proposed approach.

The paper is organized as follows. The formulation of an unsteady problem for a convection and a space-fractional diffusion equation is given in Section 2. Finite element approximations in space is discussed in Section 3. In Section 4, we construct a special additive difference scheme for time and investigate its stability. The results of numerical experiments are described in Section 5.

2. Problem formulation

In a bounded polygonal domain $\Omega \subset R^m$, $m = 1, 2, 3$ with the Lipschitz continuous boundary $\partial\Omega$, we search the solution for a comvection-diffusion problem. The diffusion process is described using a fractional power of an elliptic operator.

Define the elliptic operator as

$$\mathcal{D}u = -\mathrm{div} k(\boldsymbol{x}) \mathrm{grad}\, u + c(\boldsymbol{x})u \tag{1}$$