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HIGH ORDER GALERKIN METHODS WITH GRADED MESHES FOR TWO-DIMENSIONAL REACTION-DIFFUSION PROBLEMS

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Abstract. We develop high-order Galerkin methods with graded meshes for solving the two-dimensional reaction-diffusion problem on a rectangle. With the help of the comparison principle, we establish upper bounds for high order partial derivatives of an arbitrary order of its exact solution. According to prior information of the high order partial derivatives of the solution, we design both implicit and explicit graded meshes which lead to numerical solutions of the problem having an *optimal* convergence order. Numerical experiments are presented to confirm the theoretical estimate and to demonstrate the outperformance of the proposed meshes over the Shishkin mesh.

Key words. Singularly perturbation, reaction-diffusion problem, priori estimates, graded meshes, Galerkin method.

1. Introduction

The singular perturbation problem is an important class of boundary value problems which have broad applications. When the perturbation parameter is sufficiently small, the solution of the problem will have significantly large (partial) derivatives near the boundary, which illustrates *boundary layers*. The existence of boundary layers brings difficulty to numerical solutions of the problem, making the standard numerical methods unstable and fail to yield accurate results (cf. [26, 32]). In order to overcome the difficulty, various special meshes were constructed for singularity perturbation problems, among which the Bakhvalov type mesh and Shishkin type mesh are frequently used. Layer-adapted meshes (cf. [18]) are growing popular. Graded meshes were investigated in [7, 22, 23]. In particular, meshes proposed in [22, 23] for the one dimensional problem based on prior estimates of high order derivatives of the exact solution lead to uniform convergent solutions with optimal convergence rates for high order singular perturbation problems. A number of numerical schemes [19, 34, 37, 39, 41, 42] were established based on special meshes, providing accurate numerical solutions to one-dimensional singular perturbation problems.

Useful meshes may be constructed according to the behavior of the true solution. As a result, it is important to estimate the derivatives of the exact solution of the problem. For one-dimensional singular perturbation problems, the solutions were characterized clearly (cf. [28]), which provides valuable information to the analysis of numerical methods. For the two-dimensional problems, estimating the high order (mixed) derivatives of the solutions is much more involved. The authors of [15] utilized the maximum principle to establish the upper bounds of partial derivatives

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(with respect to a single variable x or y) of the solution of two-dimensional reaction-diffusion problems. With the aid of the estimates, the convergence properties of the numerical methods based on the Shishkin meshes were analyzed in [13, 14]. Asymptotic expansion is another tool to investigate the behavior of the solution. In [3], an asymptotic expansion of the solution was constructed, which contains boundary layer terms for edges and corner layer terms for vertices of the domain, and a uniform bound for the remainder was established to validate the uniform convergence of the expansion. The authors of [8] used the Btuzov expansion to expand the partial derivatives of the solution in a form that shows explicitly both the traditional corner singularities and boundary layers of the solution. In [21] the solution was decomposed into four terms, two of which were estimated pointwise and the other two were estimated with respect to a function norm. Based on knowledge of the solution of the singular perturbation problem on a rectangle, several popular numerical methods including the finite difference methods [16, 25, 35], the standard finite element methods [33, 38, 40] and the streamline diffusion finite element methods [20, 27, 36] were discussed. The finite volume methods [2, 10] and some adaptive schemes [4, 29] were also considered. More results can be found in [24] and the surveys [11, 12].

The goal of this paper is to construct numerical schemes of high accuracy for solving the reaction-diffusion problem on a rectangle based on prior information of high order derivatives of its exact solution. For this purpose, we present pointwise estimates of the high order (mixed) partial derivatives of the solution. Most existing results in the literature estimate the derivatives with respect to special function norms, and pointwise estimates are presented only for derivatives of order lower than two. Although the knowledge of mixed partial derivatives is not necessary in all circumstances, it is definitely crucial for some numerical methods such as sparse grid schemes. For both theoretical interest and computational purpose, we establish pointwise estimates for all types of derivatives of arbitrary orders. The upper bounds we give in this paper have a unified form, which illustrates a comprehensive view of the solution. The tool we use to establish the upper bounds is the maximum principle with a help of the solutions of related auxiliary problems.

The upper bounds of the exact solution of the problem suggest using the tensor product form of the one dimensional meshes to construct our approximate solutions. There are meshes leading to numerical solutions with optimal convergence rate for one-dimensional problems [7, 22, 23, 31]. Nevertheless, the meshes were constructed *implicitly*, which may bring inconvenience to the implementation of numerical schemes. We propose an *explicit realization* of the mesh from [22]. The solutions of the numerical methods based on the explicit realization are proved to converge at optimal order. Numerical results are presented to demonstrate the theoretical estimates and compare the proposed graded mesh with the well-known Shishkin mesh. The numerical results show that the proposed graded mesh outperforms the Shishkin mesh especially for high-order elements.

The paper is organized in four sections plus an appendix. In section 2 we present the upper bounds of the (mixed) partial derivatives of the solution. We describe in section 3 the Galerkin methods associated with graded rectangular meshes, the solutions of which converge uniformly at optimal rates. The positions of the knots are identified explicitly. In section 4 we present numerical examples to demonstrate the theoretical estimates on convergence of the numerical solutions. In the appendix, we provide details of proofs of two technical results necessary for establishing the upper bounds of the mixed partial derivatives of the exact solution.