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ANALYSIS OF OPTIMAL ERROR ESTIMATES AND SUPERCONVERGENCE OF THE DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS IN ONE SPACE DIMENSION

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Abstract. In this paper, we study the convergence and superconvergence properties of the discontinuous Galerkin (DG) method for a linear convection-diffusion problem in one-dimensional setting. We prove that the DG solution and its derivative exhibit optimal $\mathcal{O}(h^{p+1})$ and $\mathcal{O}(h^p)$ convergence rates in the L^2-norm, respectively, when p-degree piecewise polynomials with $p \geq 1$ are used. We further prove that the p-degree DG solution and its derivative are $\mathcal{O}(h^{2p})$ superconvergent at the downwind and upwind points, respectively. Numerical experiments demonstrate that the theoretical rates are optimal and that the DG method does not produce any oscillation. We observed optimal rates of convergence and superconvergence even in the presence of boundary layers when Shishkin meshes are used.

Key words. Discontinuous Galerkin method, convection-diffusion problems, singularly perturbed problems, superconvergence, upwind and downwind points, Shishkin meshes.

1. Introduction

Problems involving convection and diffusion arise in several important applications throughout science and engineering, including fluid flow, heat transfer, among many others. Their typical solutions exhibit boundary and/or interior layers. It is well-known that the standard continuous Galerkin finite element method exhibits poor stability properties for singularly perturbed problems. One of the difficulties in numerically computing the solution of singularly perturbed problems lays in the so-called boundary layer behavior. In the presence of sharp boundary or interior layers, nonphysical oscillations pollute the numerical solution throughout the computational domain. In other words, the solution varies very rapidly in a very thin layer near the boundary. Consult [49, 59, 58, 40, 55, 43] and the references cited therein for a detailed discussion on the topic of singularly perturbed problems. The discontinuous Galerkin (DG) methods have become very popular numerical techniques for solving ordinary and partial differential equations. They have been successfully applied to hyperbolic, elliptic, and parabolic problems arising from a wide range of applications. Over the last years, there has been much interest in applying the DG schemes to problems where the diffusion is not negligible and to convection-diffusion problems.

The DG method considered here is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. DG method combines many attractive features of the classical

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finite element and finite volume methods. It is a powerful tool for approximating some differential equations which model problems in physics, especially in fluid dynamics or electrodynamics. Comparing with the standard finite element method, the DG method has a compact formulation, *i.e.*, the solution within each element is weakly connected to neighboring elements. DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [46]. In 1974, LaSaint and Raviart [42] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ordinary differential equations [7, 23, 41, 42], hyperbolic [19, 20, 21, 22, 34, 35, 45, 38, 39, 30, 57, 44, 2, 3, 16, 6] and diffusion and convection-diffusion [17, 18, 53, 36] partial differential equations. The proceedings of Cockburn *et al.* [33] an Shu [51] contain a more complete and current survey of the DG method and its applications.

In recent years, the study of superconvergence of numerical methods has been an active research field in numerical analysis. Superconvergence properties for finite element and DG methods have been extensively studied in [7, 11, 37, 42, 56, 52] for ordinary differential equations, [2, 3, 16, 6, 4, 15, 13, 7, 10] for hyperbolic problems and [14, 5, 9, 10, 16, 24, 27, 30] for diffusion and convection-diffusion problems, just to mention a few citations. A knowledge of superconvergence properties can be used to (i) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (ii) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. Typically, *a posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree is raised (p-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [12].

The first superconvergence result for standard DG solutions of hyperbolic PDEs appeared in Adjerid *et al.* [7]. The authors showed that standard DG solutions of one-dimensional hyperbolic problems using p-degree polynomial approximations exhibit an $\mathcal{O}(h^{p+2})$ superconvergence rate at the roots of $(p+1)$-degree Radau polynomial. They further established a strong $\mathcal{O}(h^{2p+1})$ superconvergence at the downwind end of every element. Recent work on other numerical methods for convection-diffusion and for pure diffusion problems has been reviewed by Cockburn *et al.* [32]. In particular, Baumann and Oden [18] presented a new numerical method which exhibits the best features of both finite volume and finite element techniques. Rivière and Wheeler [47] introduced and analyzed a locally conservative DG formulation for nonlinear parabolic equations. They derived optimal error estimates for the method. Rivière *et al.* [48] analyzed several versions of the Baumann and Oden method for elliptic problems. Wihler and Schwab [54] proved robust exponential rates of convergence of DG methods for stationary convection-diffusion problems in one space dimension. We also mention the work of Castillo, Cockburn, Houston, Süli, Schötzau and Schwab [50, 25, 26] in which optimal a priori error estimates for the hp-version of the local DG (LDG) method for convection-diffusion problems are investigated. Later Adjerid *et al.* [8, 9] investigated the superconvergence of the LDG method applied to diffusion and transient convection-diffusion problems. More recently, Celiker and Cockburn [27] proved a new superconvergence property of a large class of finite element methods for one-dimensional steady state convection-diffusion problems. We also mention the recent work of Shu *et al.*