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NEW DEVELOPMENTS ON THE COUPLING OF MIXED-FEM AND BEM FOR THE THREE-DIMENSIONAL EXTERIOR STOKES PROBLEM

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Abstract. In this manuscript we consider the three dimensional exterior Stokes problem and study the solvability of the corresponding continuous and discrete formulations that arise from the coupling of a dual-mixed variational formulation (in which the velocity, the pressure and the stress are the original main unknowns) with the boundary integral equation method. The present work is an extended and completed version of the analysis and results provided in our previous paper [ZAMM Z. Angew. Math. Mech. 93 (2013), no. 6-7, 437–445]. More precisely, after employing the incompressibility condition to eliminate the pressure, we consider the resulting velocity-stress-vorticity approach with different kind of boundary conditions on an annular bounded domain, and couple the underlying equations with either one or two boundary integral equations arising from the application of the usual and normal traces to the Green representation formula in the exterior unbounded region. As a result, we obtain saddle point operator equations, which are then analyzed by the well-known Babuška-Brezzi theory. We prove the well-posedness of the continuous formulations, identifying previously the space of solutions of the associated homogeneous problem, and specify explicit hypotheses to be satisfied by the finite element and boundary element subspaces in order to guarantee the stability of the respective Galerkin schemes. In particular, following a similar analysis given recently for the Laplacian, we are able to extend the classical Johnson & Nédélec procedure to the present case, without assuming any restrictive smoothness requirement on the coupling boundary, but only Lipschitz-continuity. In addition, and differently from known approaches for the elasticity problem, we are also able to extend the Costabel & Han coupling procedure to the 3D Stokes problem by providing a direct proof of the required coerciveness property, that is without argueing by contradiction, and by using the natural norm of each space instead of mesh-dependent norms. Finally, we briefly describe concrete examples of discrete spaces satisfying the aforementioned hypotheses.

Key words. Mixed-FEM, BEM, 3D Stokes problem, Johnson & Nédélec's coupling, Costabel & Han's coupling.

1. Introduction

The classical approach combining finite element (FEM) with boundary element methods (BEM) for solving exterior boundary value problems in continuum mechanics, usually known as the coupling of FEM and BEM, has been extensively employed since its creation during the second half of the seventies up to nowadays. The usual procedure is as follows. The underlying domain is first divided into two subregions by introducing an auxiliary boundary Γ, if necessary, so that the original exterior problem can be reformulated as a transmission problem through Γ. Next, the latter is reduced to an equivalent problem in the bounded inner region by imposing nonlocal boundary conditions on Γ that are derived by employing boundary integral equation methods in the unbounded outer domain. The resulting nonlocal

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boundary value problem is then solved by a conventional Galerkin method, in which the boundary integral operators involved are discretized using finite element spaces on Γ.

While detailed surveys on most of the different ways of coupling BEM and FEM can be seen in [43] and [27, Chapter I], we simply recall here that the most popular ones correspond to the *Johnson & Nédélec* (J & N) and *Costabel & Han* (C & H) procedures (cf. [12], [13], [20], [38], [45], and [58]), which employ the Green representation of the solution in the unbounded region. The success of the J & N method, being based on a single boundary integral equation on Γ and the Fredholm theory, hinged on the fact that certain boundary integral operators are compact, which usually requires Γ to be smooth enough. According to it, it was not possible, at least from a theoretical point of view, to employ this approach when the coupling boundary was non–smooth, say for instance polygonal, which left out the possibility of utilizing classical finite element discretizations. Moreover, the J & N idea seemed to be applicable only to the Laplace operator since for other elliptic systems, such as the elasticity one, and irrespective of the smoothness of the boundaries, the aforementioned compactness did not hold. One attempt to overcome this was suggested in [9] where the underlying transmission problem was replaced by one employing the pseudostress instead of the usual stress. As a consequence, the foregoing mapping property was achieved, but the coupling boundary was still required to be smooth enough. One has to admit, however, that the above described drawbacks were mainly theoretical since no failure of the corresponding discrete schemes was ever reported by users of the method in problems where those hypotheses were not met. Any way, in order to circumvent these apparent difficulties, suitable modifications of the original J & N method, in which neither the compactness nor the smoothness play any role, were proposed by *Costabel* and *Han* in [20] and [38], respectively. Both techniques are based on the addition of a boundary integral equation for the normal derivative (resp. traction in the case of elasticity). The former leads to a symmetric and non-positive definite scheme, while the latter, on the contrary, yields a positive definite and non-symmetric scheme. Nevertheless, and since the only difference between these formulations lies on the sign of an integral identity, from now on we simply refer to either one of them as the C & H approach. Further and later contributions in this direction, including applications to nonlinear problems and coupling with mixed-FEM, non–conforming FEM, local discontinuous Galerkin, and hybridizable discontinuous Galerkin methods, can be found in [8], [14], [15], [16], [17], [18], [19], [21], [24], [25], [26], [34], [35], [36], [49], and the references therein.

The whole picture on the coupling of FEM and BEM, and particularly the widely accepted fact since the eighties concerning the lack of further applicability and usefulness of the J & N method, changed dramatically with [52]. More precisely, it was proved in this paper, without any need of applying Fredholm theory nor assuming smooth domains, that all Galerkin methods for this approach are actually stable, thus allowing the coupling boundary Γ to be polygonal/polyhedral. As a consequence, the classical J & N method was begun to be considered as a real competitor of the C & H approach. In other words, the appearing of [52] gave rise to several new contributions within this and related topics. Indeed, we first refer to [48] where the corresponding extension to the combination of mixed-FEM and BEM on any Lipschitz-continuous interface Γ was successfully developed. Furthermore, the analysis of the quasi–symmetric procedure from [9] was improved in [30] by showing that the interface Γ can also be taken polygonal/polyhedral, and that in