INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 4, Pages 525–544

WEAK GALERKIN FINITE ELEMENT METHOD FOR SECOND ORDER PARABOLIC EQUATIONS

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Abstract. We apply in this paper the weak Galerkin method to the second order parabolic differential equations based on a discrete weak gradient operator. We establish both the continuous time and the discrete time weak Galerkin finite element schemes, which allow using the totally discrete functions in approximation space and the finite element partitions of arbitrary polygons with certain shape regularity. We show as well that the continuous time weak Galerkin finite element method preserves the energy conservation law. The optimal convergence order estimates in both H^1 and L^2 norms are obtained. Numerical experiments are performed to confirm the theoretical results.

Key words. Weak Galerkin finite element methods, discrete gradient, parabolic equations.

1. Introduction

The weak Galerkin (WG for short) finite element method refers to the finite element techniques for partial differential equations where the differential operators (e.g., gradient, divergence, curl, Laplacian) are approximated by weak forms. In [17], a WG method was introduced and analyzed for second order elliptic equations based on a discrete weak gradient arising from RT element [13] or BDM element [1]. When using the RT or the BDM element, the WG finite element method requires the classical finite element partitions such as triangles for 2-dimensional elements and tetrahedra for 3-dimensional elements, which limits the use of the newly-developed method. This problem was first dealt with in [16] where for the second order elliptic equation the authors using the stabilization for the flux variable established a WG mixed finite element method that is applicable for general finite element partitions consisting of shape regular polytopes (e.g., polygons in 2D and polyhedra in 3D). The idea of stabilization has been applied to the Galerkin finite element method for the second order elliptic equation, see [10]. At present, the WG method has attracted many attentions and successfully found its way to many applications, for example, see [12] for the Helmholtz equation and [11] for elliptic interface problems.

As far as the parabolic problem is concerned, there are surely many classical numerical methods applicable. For example, see [5, 3] for the classical finite element methods, [6, 9] for the discontinuous Galerkin finite element methods, [19, 2, 4, 14, 8] for the finite volume methods. We note here that the WG method is also applicable for such kind of time dependent problems. In [7], the authors discussed the WG finite element method for the parabolic equations, where again the definition of the discrete weak gradient operator proposed in [17] was applied. Comparing to the existing methods, the WG finite element method allows using discontinuous function space as the approximation space and thus it is not necessarily to require the underlying solutions to be smooth enough as in the usual sense. This property makes the WG finite element method more flexible in applications.

Received by the editors February 15, 2015 and, in revised form, September 30, 2015.
2000 *Mathematics Subject Classification.* 65M60, 65M12.

The goal of this paper is, different from the technique applied in [7], to apply the WG finite element method to the parabolic partial differential equations, by using the idea of stabilization. We consider the following initial-boundary value problem for the second order parabolic equations

$$u_t - \nabla\cdot(a\nabla u) = f, \quad \text{for} \quad x\in\Omega, t\in J, \tag{1}$$

$$u = 0, \quad \text{for} \quad x\in\partial\Omega, t\in J, \tag{2}$$

$$u(\cdot,0) = \psi, \quad \text{for} \quad x\in\Omega, \tag{3}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary, $J = (0,T]$ with $T>0$, $a = a(\cdot)_{2\times 2}\in[L^\infty(\Omega)]^{2\times 2}$ is a symmetric matrix-valued function. Assume that the matrix function $a(\cdot)$ satisfies the following property: there exist two constants $0<\bar{\alpha}_1<\bar{\alpha}_2$ such that

$$\bar{\alpha}_1\xi^T\xi \le \xi^T a\xi \le \bar{\alpha}_2\xi^T\xi, \qquad \forall\xi\in\mathbb{R}^2. \tag{4}$$

The standard variational form for (1)-(3) seeks $u(\cdot,t)\in H_0^1(\Omega)$ such that

$$\begin{aligned}(u_t,v)+(a\nabla u,\nabla v) = (f,v), \quad \forall v\in H_0^1(\Omega), t\in J.\\ u(\cdot,0)=\psi,\end{aligned} \tag{5}$$

where $(\cdot,\cdot)$ denotes the L^2-inner product.

In this paper, we, based on the definition of a discrete weak gradient operator proposed in [10], derive the continuous time and the discrete time WG finite element methods for problems (1)-(3) by taking into account the idea of stabilization. The new obtained methods allow the application of the more general finite element partitions satisfying certain shape regular conditions, and allow as well using totally discontinuous function space as the approximation space. In addition, the continuous time WG finite element method preserves the energy conservation law.

The rest of this paper is organized as follows. In Section 2, we introduce the notations and establish the continuous time and the discrete time WG finite element schemes for problems (1)-(3). We then prove the energy conservation law of the continuous time WG approximation in Section 3. In Section 4, the optimal error estimates in both H^1 norm and L^2 norm are proved. Finally, we present the numerical example to verify the theory.

2. The WG approximation

To introduce the WG finite element method for the parabolic equations (1)-(3), we need to consider first the weak gradient and the discrete weak gradient. The gradient ∇ is a principle differential operator involved in the variational form. Thus, it is critical to define and understand discrete weak gradients for the corresponding numerical methods. Following the idea in [17, 10], the discrete weak gradient is given by approximating the weak functions with piecewise polynomial functions, as shown in what follows.

2.1. Weak gradient. Let $K\subset\Omega$ be any polygonal domain with boundary ∂K. A weak function on the region K refers to a generelized function $v=\{v_0,v_b\}$ such that $v_0\in L^2(K)$ and $v_b\in H^{1/2}(\partial K)$. The first component v_0 can be understood as the value of v in K, and the second component v_b represents the value of v on the boundary ∂K. Note that v_b may not necessarily be related to the trace of v_0 on ∂K, if it is well-defined. Denote the space of weak functions on K by

$$W(K) := \{v=\{v_0,v_b\} : v_0\in L^2(K), v_b\in H^{1/2}(\partial K)\}. \tag{6}$$