INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 4, Pages 545–568

ON DISCONTINUOUS FINITE VOLUME APPROXIMATIONS FOR SEMILINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this article, we discuss and analyze discontinuous finite volume approximations of the distributed optimal control problems governed by a class of semilinear parabolic partial differential equations with control constraints. For the spatial discretization of the state and costate variables, piecewise linear elements are used and an implicit finite difference scheme is used for time derivatives; whereas, for the approximation of the control variable, three different strategies are used: variational discretization, piecewise constant and piecewise linear discretization. *A priori* error estimates (for these three approaches) in suitable L^2-norm are derived for state, co-state and control variables. Numerical experiments are presented in order to assure the accuracy and rate of the convergence of the proposed scheme.

Key words. Semilinear parabolic optimal control problems, variational discretization, piecewise constant and piecewise linear discretization, discontinuous finite volume methods, *a priori* error estimates, numerical experiments.

1. Introduction

1.1. Scope. The purpose of this paper is to introduce discontinuous finite volume methods for the approximations of control, state and co-state variables involved in a semilinear parabolic optimal control problems. As it is well known that optimal control problems governed by a class of partial differential equations (introduced in [20], [34]) have various applications in scientific and engineering-related problems. For instance, heat conduction, diffusion, electromagnetic waves, fluid flows, freezing processes, and many other physical phenomena can be put forward as models based on partial differential equations. In particular, parabolic optimal control problems are used in describing a controlled heat transfer process for optimal cooling of steel profiles. The optimization of semilinear heat equations represent mathematical model for many physical applications, e.g. laser hardening, welding of steel, laser thermotherapy (used for cancer treatment) etc.

Due to the computational simplicity, efficiency and robustness of finite element methods, these methods are extensively employed for the approximation of optimal control problems. For instance, the finite element error analysis for elliptic optimal control problems has been established in [8, 9, 13, 32, 35] and references therein. In addition to that finite element approximations for parabolic optimal control problems have been discussed in [26, 27, 33, 36] and references cited in these articles. In most of these articles, the state and costate variables were approximated by conforming (continuous) finite element methods in which piecewise linear polynomials are used and control variable by piecewise constant or piecewise linear polynomials. For control variable, the rate of convergence is of $\mathcal{O}(h)$ and $\mathcal{O}(h^{3/2})$ for piecewise constant and piecewise linear discretization, respectively. For discretization of the

Received by the editors January 25, 2015 and, in revised form, November 24, 2015.
1991 *Mathematics Subject Classification.* 65Q05.

control which is the primary variable, a variational approach is proposed by Hinze in which control set is not discretized explicitly but discretized by a projection (to be defined later) and obtained improved convergence of $\mathcal{O}(h^2)$, for more details, we refer to [14].

The typical inter-element continuity criteria which is usually imposed on finite dimensional trial spaces involved in conforming and even nonconforming finite element methods in order to make the resulting system well-posed is no longer required for discontinuous Galerkin (DG) methods. Apart from this, other intrinsic attractive features of DG methods are: suitability for local mesh adaptivity, element-wise conservative, they allow high degree polynomials in different elements and can easily handle non-standard boundary conditions. For more details regarding DG methods, we refer to [1, 2, 30, 31] and references therein. In the context of control problems, DG methods have also been employed for parabolic optimal control problems, for instance see [26, 27, 29].

On the other hand, finite volume element (FVE) methods can be considered as Petrov-Galerkin methods in which the finite dimensional trial space consists of piecewise linear polynomials and piecewise constant functions are used in the test space. We can expect the computational advantages of FVE methods over finite element methods, as the test space associated with the dual grid is piecewise constant. In addition, the desirable feature of FVE methods is conservation of a quantity of interest, e.g., mass, momentum or energy. Due to this property of local conservation, finite volume element methods are widely used in computational fluid dynamics. However, the low regularity used in the test space, demands high regularity on given data or exact solution in order to achieve optimal L^2-estimates. For instance, for non-homogeneous elliptic problems, derivation of optimal L^2-estimates requires either an exact solution in H^3 or a source term globally in H^1 (see e.g. [12]). For more details and advantages of FVE methods, kindly see the early work [7, 10] and the recent review [19]. Recently, FVE methods have been employed in [24, 25] for the approximation of the state and costate variables appeared in linear elliptic and parabolic problems. In these articles, for discretization of the control variable, a variational discretization approach is used and optimal order of convergence has been shown.

In order to make use of desirable properties of DG methods and FVE methods, we will focus on a hybrid scheme discontinuous finite volume methods (DFVM) for the approximation of the distributed parabolic semilinear optimal control problems. These methods were originally introduced by [38] for elliptic problems and later with some modifications these methods were applied to elliptic, Stokes and parabolic problems and fluid flow problems, see [4, 6, 16, 17, 22, 37, 40, 39]. Recently, in [18] Kumar proposed a stabilized DFVM formulation for more general Stokes problems. However, up-to to our knowledge, there are hardly any results available on DFVM for the approximation of semilinear parabolic optimal control problems. Therefore, in this article an attempt has been made to introduce a fully discrete discontinuous finite volume methods for the approximation of the parabolic control problems. In addition, we use three different approaches: variational discretization (introduced in [14]), piecewise linear and piecewise constant discretization to approximate the control.

For the solvability of the optimal control problems, in literature, there are two different approaches: one is *discretize-then-optimize* and another one is *optimize-then-discretize*. In the *discretize-then-optimize* approach, one first discretizes the