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IMPROVED ADI PARALLEL DIFFERENCE METHOD FOR QUANTO OPTIONS PRICING MODEL

XIAOZHONG YANG, FAN ZHANG, AND LIFEI WU

Abstract. The quanto options pricing model is a typical two-dimensional Black-Scholes equation with a mixed derivative term, and it has been increasingly attracting interest over the last decade. A kind of improved alternating direction implicit methods, which is based on the Douglas-Rachford (D-R ADI) and Craig-Sneyd (C-S ADI) split forms, is given in this paper for solving the quanto options pricing model. The improved ADI methods first split the original problem into two separate one-dimensional problems, and then solve the tri-diagonal matrix equations at each time-step. There are several advantages in this method such as: parallel property, unconditional stability, convergency and better accuracy. The numerical experiments show that this kind of methods is very efficient compared to the existent explicit finite difference method. In addition, because of the natural parallel property of the improved ADI methods, the parallel computing is very easy, and about 50% computational cost can been saved. Thus the improved ADI methods can be used to solve the multi-asset option pricing problems effectively.

Key words. Quanto options pricing model, two-dimensional Black-Scholes equation, improved ADI method, parallel computing, numerical experiments.

1. Introduction

In today's financial market, option is one of the most important financial derivatives. With the rapid development of financial market, it is difficult to meet the needs of the financial traders by only using European, American and other single asset options. Therefore, the financial institution designs many multi-asset options.

The name "quanto" is, in fact, derived from the variable notional amount, and is short for "quantity adjusting options". A quanto is a type of derivative in which the underlying is denominated in one currency, but the instrument itself is settled in another currency at some fixed rate. Generally, the value of the quanto options depends not only on the option's intrinsic value in the foreign currency, but also on the foreign currency exchange rate. Therefore, the qunato options pricing is relatively complex. This paper is mainly devoted to the two-dimensional (2D) Black-Scholes equation of quanto options pricing model [1, 2, 3]

$$\frac{\partial V}{\partial t} + F_1(V, S_1, S_2) + G_1(V, S_1, S_2) - r_1 V = 0,$$

$$F_1(V, S_1, S_2) = \frac{1}{2}[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}],$$

$$G_1(V, S_1, S_2) = (r_1 - \widehat{q_1}) S_1 \frac{\partial V}{\partial S_1} + (r_1 - \widehat{q_2}) S_2 \frac{\partial V}{\partial S_2},$$

$$\widehat{q_1} = r_1 - r_2 + q + \rho\sigma_1\sigma_2,$$

$$\widehat{q_2} = r_2.$$

Here, V, S_1, S_2, r_1, r_2, σ_1^2, σ_2^2, ρ, q and T are the price of the quanto options, price of foreign risk asset, exchange rate of foreign currency against domestic one, domestic risk-free rate, foreign risk-free rate, variance of the rate of return on S_1,

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variance of the rate of return on S_2, correlation coefficient, interest rate and time to expiration, respectively. The quanto options pricing model has the following analytic solution (cf. [1, 2, 3])

$$V(S_1,S_2,t)=\frac{1}{2\pi(T-t)}e^{-r_1(T-t)}\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}H(S_1,S_2,t), \tag{1}$$

$$H(S_1,S_2,t)=\int_0^\infty\int_0^\infty\frac{(\eta_1-K)^+}{\eta_1}exp[-\frac{\sigma_2^2\alpha_1^2-2\rho\sigma_1\sigma_2\alpha_1\alpha_2+\sigma_1^2\alpha_2^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)(T-t)}]d\eta_1 d\eta_2,$$
$$\alpha_1=\ln\frac{S_1}{\eta_1}+(r_2-q-\rho\sigma_1\sigma_2-\frac{\sigma_1^2}{2})(T-t),$$
$$\alpha_2=\ln\frac{S_2}{\eta_2}+(r_1-r_2-\frac{\sigma_2^2}{2})(T-t).$$

Although the quanto options pricing model has an analytic solution (1), it cannot satisfy the application requirements due to the computing complexity of the expression (1). For more general settings, we usually use the numerical method instead, such as the Monte-Carlo method [4] and Binomial Tree method (cf.[5]), but because a large amount of numerical simulations are needed for getting a high accuracy, the calculation efficiency of the above methods are not better than that of the finite difference method, which consists of replacing the partial derivatives by numerical differentiation and then solving the resulting discretized problem.

So far, the finite difference methods used for solving the option pricing problems have got a lot of progresses. M. Gilli *et al.*(2002) [6] investigated an explicit-implicit difference scheme for multi-asset option pricing model, but the solution of a large block triangular linear system was required and the calculation was relatively complex in this method; X.Z. Yang *et al.*(2007) [7] proposed a general difference scheme for solving the one-dimensional Black-Scholes equation, but they did not consider the multi-dimensional ones; A.Q.M. Khaliq *et al.*(2008) [8] put forward a new finite difference method for solving the 2D Black-Scholes equation, but it also needed the using of penalty approach method which was not very convenient to calculate by computers; X.Z. Yang and G.X. Zhou (2011) [9] used the additive operator splitting(AOS) method for solving the quanto options pricing model, but because the approximation to the mixed derivative term was not efficient, the accuracy of this method was not very well. R.Company *et al.* (2008)[10] and D.Y.Tangman (2008) [11] put forward high-order finite difference schemes for solving the nonlinear Black-Scholes equation, but the computational efficiency was not very well.

In the numerical analysis, the alternating direction implicit(ADI) method is mostly notable for solving the partial differential equation in two or more dimension (cf. [12, 13, 14]). I.J.D. Craig and A.D. Sneyd(1988) [15] first put forward an ADI scheme for N-dimensional parabolic equations with a mixed derivative term, but the scheme was conditionally stable and less effective; S. McKee *et al.*(1996) [16] introduced a new ADI scheme which was capable of solving a general parabolic equation in two dimension with mixed derivative and convective terms and was proved to be unconditionally stable, but the equation they considered did not include the one degree term; In addition, D. Jeong, J. Kim(2013) [17] used the ADI difference scheme on multi-dimensional Black-Scholes option pricing models, however the scheme was not unconditionally convergent. Similar researches also have been done for solving the parabolic equations and other types of equations(cf. [22, 23, 24, 25, 32, 33]). For these reasons, this paper gives an improved ADI difference scheme that is capable of solving the quanto options pricing model with unconditional stability and good convergency properties.