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# A PRIORI ERROR ESTIMATES OF A SIGNORINI CONTACT PROBLEM FOR ELECTRO-ELASTIC MATERIALS

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**Abstract.** We consider a mathematical model for a static process of frictionless unilateral contact between a piezoelectric body and a conductive foundation. A variational formulation of the model, in the form of a coupled system for the displacements and the electric potential, is derived. The existence of a unique weak solution for the problem is established. We use the penalty method applied to the frictionless unilateral contact model to replace the Signorini contact condition, we show the existence of a unique solution, and derive error estimates. Moreover, under appropriate regularity assumptions of the solution, we have the convergence of the continuous penalty solution as the penalty parameter $\epsilon$ vanishes. Then, the numerical approximation of a penalty problem by using the finite element method is introduced. The error estimates are derived and convergence of the scheme is deduced under suitable regularity conditions.

**Key words.** Piezoelectric, variational inequality, Signorini condition, penalty method, fixed point process, finite element approximation, error estimates.

## 1. Introduction

In recent years, piezoelectric materials have triggered intensive studies to fulfill their potential applications in a variety of fields due to include the coupling between the mechanical and electrical material properties. Indeed, there is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, see, e.g., [1, 2, 4, 7, 8, 9, 11] and the references therein. Here, we consider a mathematical model which describes the frictionless contact between an piezoelectric body and a foundation, within the framework of small deformations theory. The material's behavior is modeled with a linear electroelastic constitutive law, the process is static and the foundation is assumed to be electrically conductive. Contact is described with the Signorini contact conditions and a regularized electrical conductivity condition. The numerical approximation of a static unilateral contact problems with or without friction for piezoelectric materials can be found in [1, 2, 5, 7].

In the present work, the numerical approximations were based on variational inequalities modeling unilateral contact in piezoelectricity. Here, a penalty method is employed to replace the Signorini contact condition. This approach was used previously by F. Chouly and P. Hild [3] to numerically approximate the solution of contact problems in linear elasticity. The novelty of the paper is in dealing with a model which couples the piezoelectric properties of the material with the electrical conductivity conditions on the contact surface. Consideration of the electrical contact condition leads to nonstandard boundary conditions on the contact surface and supplementary nonlinearities in the problem. Because of the latter and piezoelectric effect, the mathematical problem is formulated as a coupled system of the variational inequality for the displacement field and non-linear variational equation for the electric potential. In this paper, We analyze both the continuous and discrete (using continuous conforming piecewise linear finite element methods)

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problems. We show that the theoretical convergence of the penalty method gives the best results when $\epsilon = h$, where $\epsilon$ is the penalty parameter, and $h$ is the mesh size. We note that the convergence is limited by the same terms involved when considering the direct approximation of the variational inequality without penalty.

The paper is organized as follows. In Section 2 we present the models of electroelastic frictionless unilateral contact with the electrical contact condition. list the assumptions on the data, derive the variational formulation of each model, and state the existence and uniqueness result. In Section 3 we introduce the penalty problem and show that it has a unique solution. In Section 4, we describe the finite element approximation of the penalty problem and we present the results of some error estimates for the numerical approximation. finally, The proof of the main result is provided in Section 5.

## 2. Setting of the problem and variational formulation

**2.1. The contact problem.** In this section we describe the problem of unilateral frictionless contact between a piezoelectric body and a conductive foundation.
The physical setting is the following : we consider an elasto-piezoelectric body which initially occupies an open bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently smooth boundary $\partial\Omega = \Gamma$. The body is acted upon by a volume forces of density $f_0$ and has volume electric charges of density $q_0$. It is also constrained mechanically and electrically on the boundary. To describe these constraints we decompose $\Gamma$ into three mutually disjoint open parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts $\Gamma_a$ and $\Gamma_b$, on the other hand, such that $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$. The body is clamped on $\Gamma_D$ and a surface tractions of density $f_2$ act on $\Gamma_N$. Moreover, the electric potential vanishes on $\Gamma_a$ and the surface electric charge of density $q_2$ is prescribed on $\Gamma_b$. On $\Gamma_C$ the body may come into contact with a conductive obstacle, the so called foundation. We assume that the foundation is electrically conductive and its potential is maintained at $\varphi_F$. The contact is frictionless unilateral and there may be electrical charges on the contact surface. The indices $i$, $j$, $k$, $l$ run between 1 and $d$. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i,j} = \partial u_i/\partial x_j$. Everywhere below we use $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$ while "$\cdot$" and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is $\forall u,\ v \in \mathbb{R}^d, \forall\sigma,\ \tau \in \mathbb{S}^d$,

$$u \cdot v = u_i \cdot v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}.$$

We denote by $u : \Omega \to \mathbb{R}^d$ the displacement field, by $\sigma : \Omega \to \mathbb{S}^d$, $\sigma = (\sigma_{ij})$ the stress tensor and by $D : \Omega \to \mathbb{R}^d$, $D = (D_i)$ the electric displacement field. We also denote $E(\varphi) = (E_i(\varphi))$ the electric vector field, where $\varphi : \Omega \to \mathbb{R}$ is an electric potential such that $E(\varphi) = -\nabla\varphi$. We shall adopt the usual notations for normal and tangential components of displacement vector and stress : $v_n = v \cdot n, \quad v_\tau = v - v_n n, \quad \sigma_n = (\sigma n) \cdot n, \quad \sigma_\tau = \sigma n - \sigma_n n$ , where $n$ denote the outward normal vector on $\Gamma$. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, and "Div", "div" denote respectively the divergence operators for tensor and vector valued functions, i.e. $\text{Div}\,\sigma = (\sigma_{ij,j}), \quad \text{div}\,D = (D_{j,j})$.

Under the previous assumption, the classical model for this process is the following.
**Problem** $P$. Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathbb{S}^d$, an