INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 5, Pages 676–688

A NEW PARALLEL FINITE ELEMENT ALGORITHM BASED ON TWO-GRID DISCRETIZATION FOR THE GENERALIZED STOKES PROBLEM

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Abstract. Based on two-grid discretization, a new parallel finite element algorithm for the generalized Stokes problem is proposed and analyzed. Motivated by the observation that for a solution to the generalized Stokes problem, low frequency components can be approximated well by a relatively coarse grid and high frequency components can be computed on a fine grid, this algorithm first solves the generalized Stokes problem on a coarse grid, and then corrects the resulted residual by standard additive Schwarz method on a fine grid. Under some regular assumptions, error estimates of the approximate solutions are provided. Numerical results are also given to illustrate the effectiveness of the algorithm.

Key words. Generalized Stokes problem, finite element, parallel algorithm, Schwarz method, two-grid method.

1. Introduction

The generalized Stokes problem arises naturally in the time discretization of non-stationary Navier-Stokes equations which mathematically model the flow motion of an incompressible Newtonian viscous fluid. It consists of the key and most time-consuming part of the solving process of time-dependent Navier-Stokes equations at each nonlinear iteration. Therefore, the development of efficient algorithms for the generalized Stokes problem is very important.

Recently, based on local finite element discretizations, an approach to local and parallel finite element computations is proposed for a class of linear and nonlinear elliptic boundary value problems in [28–30]. Based on this approach, some new local and parallel algorithms have been proposed and analyzed for the steady Stokes equations [11, 20], the stationary Navier-Stokes equations [9, 10, 14, 21], the stream function form of Navier-Stokes equations [15], and the transient Stokes equations [22]. These algorithms have low communication complexity. They only require existing sequential solver as subproblems solver and hence allow existing sequential PDE codes to run in a parallel environment with a little investment in recoding.

However, based on our analysis and numerical tests, there is still room to improve some of the above mentioned algorithms for some incompressible flow problems. First and foremost, although the coarse grid size is suitably chosen, the finite element approximations obtained from the algorithms are much less precise for some problems compared with the global standard Galerkin finite element solution, especially, when the overlapping size of subdomains is small. Secondly, the accuracy

Received by the editors received June 7, 2015 and, in revised form, July 27, 2016.
2000 *Mathematics Subject Classification.* 65N30, 65N55, 76D07, 76M10.
This research was supported by the Natural Science Foundation of China (Nos. 11361016, 11271298 and 11362021), the Project Sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry, the Fundamental Research Funds for the Central Universities (No. XDJK2014C160), the Basic and Frontier Research Program of Chongqing Municipality, China (No. cstc2016jcyjA0809), and the China Postdoctoral Science Foundation (No. 20070421155, 200801448).

of the finite element approximations not only depends on the overlapping size of subdomains, but also depends on the shapes of subdomains and hence depends on the way to decomposing the solution domain into subdomains. Finally, the approximate solutions are piecewise defined and hence are generally discontinuous, making the algorithms not applicable for these problems that continuity of the solutions is required.

The motivation of this work is to overcome the above mentioned weakness of the local and parallel algorithms and present a new improved algorithm for the generalized Stokes problem. Based on our understanding of the local and global properties of a finite element solution to the generalized Stokes problem, i.e., the global behavior of a solution to the generalized Stokes problem is mostly governed by low frequency components while local behavior is mostly governed by high frequency components, we first approximate the low frequency components of the solution on a coarse grid, then use a standard additive Schwarz method on a fine grid to correct the resulted residual (which contains mostly high frequencies). This new algorithm is an improvement of the parallel algorithm proposed in [11] for the steady Stokes problem in the sense that continuous and more precise solutions can be obtained; see Section 3.

It is noted that unlike the standard multigrid and domain decomposition methods where the two-grid method is used to devise iterative methods for solving a given discretization scheme (see, e.g., Bank [4], Hackbusch [8], Simth, Bjørstad and Gropp [23] , Quarteroni and Valli [18], Toselli and Widlund [25]), our algorithm is to design a discretization scheme. Moreover, in our algorithm, the global coarse grid problem needs to be solved only once and it does not have to be coupled with the subsequent parallel solvers.

The rest of this paper is organized as follows. In the next section, the generalized Stokes problem and its mixed finite element approximations are provided. In Section 3, a parallel algorithm based on local finite element computations proposed in [11] is reviewed. Analysis of improvement for this parallel algorithm is performed and a new improved algorithm is devised and analyzed. In Section 4, two numerical tests are carried out to illustrate the effectiveness of the new algorithm. Finally, conclusions are drawn in Section 5.

2. The generalized Stokes problem and its mixed finite element approximations

Let Ω be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ in R^d ($d = 2, 3$). We shall use the standard notations for Sobolev spaces $W^{s,p}(\Omega)$, $W^{s,p}(\Omega)^d$ and their associated norms and seminorms; see, e.g., [1, 5]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega), H^s(\Omega)^d = W^{s,2}(\Omega)^d$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, $\| \cdot \|_{s,\Omega} = \| \cdot \|_{s,2,\Omega}$. For a subdomain $\Omega_0 \subset \Omega$, we view $H_0^1(\Omega_0)$ as a subspace of $H_0^1(\Omega)$ by extending the functions in $H_0^1(\Omega_0)$ to be functions in $H_0^1(\Omega)$ with zero outside of Ω_0.

We consider the following generalized Stokes problem

$$\alpha u - \nu \Delta u + \nabla p = f \quad \text{in } \Omega, \tag{1}$$

$$\text{div } u = 0 \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $u = (u_1, \cdots, u_d)$ is the velocity, p the pressure, $f = (f_1, \cdots, f_d)$ the prescribed body force, ν the kinematic viscosity and α a positive parameter proportional to the inverse of time-step size.