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A CONSERVATIVE ENFORCING POSITIVITY-PRESERVING ALGORITHM FOR DIFFUSION SCHEME ON GENERAL MESHES

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Abstract. For a class of diffusion schemes not satisfying the property of positivity-preserving, we propose an enforcing positivity algorithm. It is locally conservative and easy to be implemented in existing codes. Moreover, this algorithm can be performed on both structured and unstructured meshes. Numerical experiments demonstrate that in terms of L_2 error and conservation this algorithm is much better than the trick of directly enforcing the negative values to zero (ENZ), which has been used in applications, meanwhile, in terms of L_∞ error it is approximate to ENZ and CEPA repairing algorithms.

Key words. positivity correction, conservation, nonlinear diffusion equation, general mesh.

1. Introduction

Diffusion process appears in many physical problems, such as the subsurface flows, radioactive material transport and inertia confinement fusion. While numerically simulating those problems, in order to obtain reliable numerical solution, it is necessary for the numerical methods to preserve some basic properties of diffusion equation such as the conservation and the maximum principle. Unfortunately, most of the discrete schemes for diffusion equations can not satisfy the discrete maximum principle(DMP) on distorted meshes, including some well-known diffusion schemes, such as the Kershaw scheme presented in [4], the MDHW scheme proposed in [10], the Nine-Point scheme (NPS) in [6], the multi-point flux approximation (MPFA) in [1], and the mimetic finite difference (MFD) methods in [7, 8]. A scheme not satisfying the DMP may produce negative numerical solution for diffusion equation with nonnegative initial value and sources term.

As it was pointed out in [14], the positivity of solutions is very important for numerical robustness. For some nonlinear diffusion problems, diffusion coefficients have no definition for negative solutions. If the numerical solutions are negative values in the process of computation, the computation procedure will break down. In order to prevent the computation procedure from interrupting, one common way is to modify the negative values by using certain repair techniques, and then the positivity-preserving can be achieved. Another way is to design positivity-preserving schemes, such as those in [2], [11], [12], [16], [21] and [22]. But, as we know, all of them with the accuracy being higher than first order are nonlinear schemes even if the diffusion equation is linear, so the computational costs will increase unavoidably.

This paper will focus on the first way of mending the "negative" cells, i.e., the numerical solutions of finite volume scheme on those cells are negative. The simplest repair technique for a posteriori correction of the discrete solution is just to

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set negative values to be zero (or a small positive number), which is easy to be implemented in existing codes but it destroys the conservation. Some improved strategies are proposed in recent years. In [9] a repair technique is proposed which enforces the linear finite element solution of elliptic equations on 2D triangular meshes to satisfy the discrete maximum principle and keep the total energy conservation. In [17] a global repair technique is devised for diamond finite volume schemes, which also keeps total energy conservation. In [5] a nonlinear constrained finite element method satisfying the discrete maximum principle to anisotropic diffusion problems is proposed, which is a positivity-preserving correction scheme. All of these repair algorithms do not maintain local conservation, i.e., they don't give consistent discrete flux on cell-edges between the neighboring cells.

In [20] a method of enforcing positivity with local conservation for nine-point schemes of nonlinear diffusion equations is developed, which allows the new solution to preserve positivity as well as local conservation at each nonlinear iteration step in solving nonlinear diffusion problems. However, this method may lead to the increase of the number of nonlinear iteration, and then computational efficiency decreases.

In this paper we develop an efficient method of enforcing negative values to zero with local conservation for nonlinear diffusion equations on both structure and unstructured meshes. It allows the new solution to preserve positivity as well as local conservation at each nonlinear iteration step .

This paper is organized as follows. In Section 2, we introduce a finite volume scheme on polygonal meshes for nonlinear diffusion equation. In Section 3, the conservative enforcing positivity-preserving algorithm is discussed. In Section 4, we give some numerical examples to verify the accuracy and conservation for our algorithm. The conclusion is summarized in Section 5.

2. The finite volume scheme on general meshes

Consider the nonlinear diffusion problem

$$\frac{\partial u}{\partial t} - \nabla \cdot (\kappa(X,t,u)\nabla u) = f(X,t), \quad X \in \Omega, \quad t \in (0,T], \tag{1}$$

$$u(X,0) = \phi(X), \quad X \in \Omega, \tag{2}$$

$$u(X,t) = \psi(X,t), \quad X \in \partial\Omega, t \in [0,T], \tag{3}$$

where $u = u(X,t)$ is a function to be solved, Ω is a polygonal domain in R^2, $X = (x,y) \in \Omega$, κ is a positive diffusion coefficient which may depend on u and could be discontinuous on Ω, and f is a given source function. The boundary conditions can also be Neumann or Robin types.

Divide Ω into polygonal meshes (see Fig. 1), in which a polygonal cell and its center are denoted by K or L. The cell edge is denoted by σ. If the σ is a common edge of cell K and L, and its vertices are A and B, then we denote $\sigma = K|L = BA$. Let $\mathcal{T}$ be the set of all cells, and $\mathcal{C}$ is the set of all cell-edges, and $\mathcal{C}_K$ is the set of all cell-edges of cell K. The length of σ is denoted by $|\sigma|$, and the area of cell K is denoted by S_K. The distance from the center of the cell K or L to the edge σ is denoted by $d_{K,\sigma}$ or $d_{L,\sigma}$ respectively. As usual we introduce a time step $\Delta t > 0$ and the time levels $t^n = n\Delta t$ with $n = 0, 1, \ldots, N$, and $t^N = T$.

By integrating (1) over the cell K and using the Green's formula, we can obtain

$$\int_K \frac{\partial u}{\partial t} dX + \sum_{\sigma \in \mathcal{C}_K} \mathcal{F}_{K,\sigma} = \int_K f(X,t)dX, \tag{4}$$