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THE UNSTABLE MODE IN THE CRANK-NICOLSON LEAP-FROG METHOD IS STABLE

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Abstract. This report proves that under the time step condition $\Delta t|\Lambda| < 1$ ($|\cdot|$ = Euclidean norm) suggested by root condition analysis and necessary for stability, all modes of the Crank-Nicolson Leap-Frog (CNLF) approximate solution to the system

$$\frac{du}{dt} + Au + \Lambda u = 0, \text{ for } t > 0 \text{ and } u(0) = u_0,$$

where $A + A^T$ is symmetric positive definite and Λ is skew symmetric, are asymptotically stable. This result gives a sufficient stability condition for non-commutative A and Λ, and is proven by energy methods. Thus, the growth, often reported in the unstable mode, is not due to systems effects and its explanation must be sought elsewhere.

Key words. IMEX method, Crank-Nicolson Leap-Frog, CNLF, unstable mode, computational mode.

1. Introduction

Implicit-explicit (IMEX) time-stepping schemes are often used for solving multi-physics problems with both stiff and nonstiff components, e.g., advection-diffusion-reaction equations, Navier-Stokes equations, geophysical flows, surface-groundwater flows. IMEX schemes treat the stiff term implicitly and the nonstiff term explicitly, and thus suffer from neither the computational expense of fully implicit schemes nor the demanding time step requirement of fully explicit methods, e.g., [1, 6, 7, 23].

The Crank-Nicolson Leap-Frog (CNLF) scheme, a classic two-step IMEX method, is frequently used in atmospheric flow simulations [1, 6, 17]. In this article, we prove asymptotic stability of the *unstable* or *computational mode* of the CNLF method for the system

$$\frac{du}{dt} + Au + \Lambda u = 0, \text{ for } t > 0 \text{ and } u(0) = u_0, \tag{1}$$

where $A_s = \frac{1}{2}(A + A^T) > 0$ (A_s is symmetric positive definite) and Λ is skew symmetric. Here $u : [0, \infty) \to \mathbb{R}^d$ and the square, *non-commutative*, real matrices A, Λ have compatible dimensions. Under these conditions, the solution to (1) satisfies $u(t) \to 0$ as $t \to \infty$, so any growth in the approximate solution is a numerics induced instability. With superscript denoting the time step number, CNLF, the IMEX combination of Crank-Nicolson and Leap-Frog, is given by: given u^0, u^1, find u^{n+1} satisfying for $n \geq 1$:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} + A\frac{u^{n+1} + u^{n-1}}{2} + \Lambda u^n = 0. \tag{CNLF}$$

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Root condition analysis of CNLF for the scalar test problem $y' + ay + i\lambda y = 0$ leads to the necessary time step condition essentially from [11]:

$$\Delta t|\Lambda| < 1, \quad |\cdot| = \text{Euclidean norm}. \tag{2}$$

This condition was recently proven (by discrete energy methods) sufficient for stability in [15].

However, in practical simulations, difficulties with CNLF's unstable mode occur. It is often reported (see for example [5], [12], [19], [2], [18], [22], [10]) that as $n \to \infty$,

$$\begin{aligned} &\text{Stable Mode: } |u^{n+1} + u^{n-1}| \to 0, \\ &\text{Unstable Mode: } |u^{n+1} - u^{n-1}| \to \infty. \end{aligned} \tag{3}$$

CNLF is used for many geophysical flow simulations from which experience with and fixes for the unstable mode are correspondingly large, e.g., [5], [12], [13], [19], [2], [18], [22], [10]. One mystery is that since CNLF is stable under (2), no growth is possible in theory and yet time filters to deal with (3) are nearly universal in practice, [10, 16, 22]. It is an open question to determine if this could be due to the gap for IMEX methods (e.g., [1], [3], [6], [8], [20], [21]) between necessary conditions from root condition analysis and sufficient ones for systems, to roundoff errors exciting the weak instability in LF not sufficiently damped by CN, to imperfect imposition of (2), to nonlinearities or other unknown causes.

We prove that under (2) *the CNLF unstable mode is asymptotically stable* for the system (1). This result, consistent with numerical tests in Section 3, supports the scenario that growth in the unstable mode is not due to a system effect but rather due to imperfect imposition of and thus slight violation of (2), or non-autonomous effects studied in [14], or the combination of roundoff errors breaking skew symmetry in Λ and near singularity of A.

Theorem 1. *Consider (CNLF) for non-commutative A, Λ. Suppose the (necessary) time step condition (2) holds. Then, all modes of CNLF are asymptotically stable:*

$$u^n \to 0 \text{ as } n \to \infty \text{ and thus both}$$
$$u^{n+1} + u^{n-1} \to 0 \text{ and } u^{n+1} - u^{n-1} \to 0.$$

Remark 2. *If the matrices A and Λ commute then this follows from standard root condition analysis. Thus, (2) is a* necessary *condition for asymptotic stability. For single linear multistep methods it is known that root conditions are also sufficient. However, for implicit-explicit combinations of different methods, such as CNLF, root conditions are* not sufficient. *For example, Asher, Ruuth and Wetton* [1] *page 811 note "these results provide necessary but not sufficient conditions for stability..." and Hundsdorfer and Ruuth* [7] *page 2019 note "Theoretical results are difficult to obtain if these linearizations do not commute...". The only general path (that we take in Section 2) to a sufficient condition for systems is through energy methods.*

2. Three examples of the structure (1)

It is very common for problems in applications to have the structure of (1), a dissipative perturbation of a conservative system. We give three simple examples.

2.1. Transport plus diffusion. Suppressing spacial discretization, suppose we take $Au = -\epsilon u_{xx}$ (typically ϵ is small). Then (1) becomes the evolutionary