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# WEIGHTED HARMONIC AND COMPLEX GINZBURG-LANDAU EQUATIONS FOR GRAY VALUE IMAGE INPAINTING

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**Abstract.** We consider two second-order variational models in the image inpainting problems. The aim is to obtain in the restored region some fine features of the initial image, e.g. corners, edges, .... The first model is a linear weighted harmonic method well suited for binary images and the second one is its extension to the complex Ginzburg-Landau equation for the inpainting of multi-gray level images. The approach that we introduce consists of constructing a family of regularized functionals and to select locally and adaptively the regularization parameters in order to capture fine geometric features of the image. The parameters selection is performed, at the discrete level, with a posteriori error indicators in the framework of the finite element method. We perform the mathematical analysis of the proposed models and show that they allows us to reconstruct accurately the edges and the corners. Finally, in order to make some comparisons with well established models, we consider the nonlinear anisotropic diffusion and we present several numerical simulations to test the efficiency of the proposed approach.

**Key words.** Image inpainting, inverse problems, regularization procedures, adaptive finite elements.

## 1. Introduction

Image inpainting (or disocclusion) refers to restoring a damaged image with missing information. This type of image processing is very important and has many applications in various fields (painted canvas, movies restoration, augmented reality, ...). In fact, many images are often scratched and damaged, and the goal in the inpainting problems is to restore deteriorated or missing parts, so that a viewer cannot distinguish them from the rest. Various mathematical and heuristic techniques were considered to address this problem, such as statistical methods [23], mathematical programing and computational geometry methods [34], we refer to the article [11] and the references therein where an exhaustive review is given for this problem and for the various approaches developed to solve it. In this article we will be concerned by the Partial Differential Equations (PDE) approach which belongs to the class of the widely used methods ([6, 12, 19, 20]). Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, denotes the entire domain of a given image $f$, the basic idea in the PDE approach, is to fill-in the damaged region $D \subset \Omega$, where the pixels of $f$ are altered or lost, by an interpolation from the available part in $\Omega \backslash D$. Usually, the PDE-based models are obtained from the mathematical knowledge of the properties of some differential operators, and aim to fulfill some a priori expectations and assumptions on the final solution. The diffusion operators are the mostly used to this end (e.g. the heat equation, the Cahn-Hilliard equation,...[10, 12, 15, 20, 28]). Usually such models are formulated as a constrained optimization problem:

$$\text{Minimize } R(u) \quad \text{given} \quad u = f + n \;\; in \;\; \Omega\backslash D, \tag{1}$$

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where the image $f$ is given in $\Omega\backslash D$ and $n$ is a Gaussian noise. $R(u)$ denotes the regularizing term, mostly a semi-norm of a functional space fixed a priori to enforce some expectations on the solution (e.g. a Sobolev space $H^s$, Bounded Variations functions space $BV, \dots$) and $u$ is the image to be reconstructed. The unconstrained formulation of (1) reads:

$$\alpha R(u) + \frac{1}{2}\int_\Omega \lambda_D (u-f)^2 dx, \tag{2}$$

where $\alpha$ is a regularization parameter and $\lambda_D = \lambda_0 \chi_{\Omega\backslash D}$ for $\lambda_0 \gg 0$, a penalization factor, and $\chi_{\Omega\backslash D}$ is the indicator function of the sub-domain $\Omega \setminus D$. These two parameters $\alpha$ and $\lambda_0$ are chosen in order to balance the regularization term $R(u)$ and the data fitting term.

Various methods use uniform parameters $\alpha$ and $\lambda$, chosen in general empirically or within the regularization theory, e.g. with Morozov's criterium when the magnitude of the noise is given [25, 31]. In many applications, this choice is not reliable and may produce the loss of some relevant features of the image such as the edges (see Fig:1). Therefore, based on the importance of the scale-space representation of

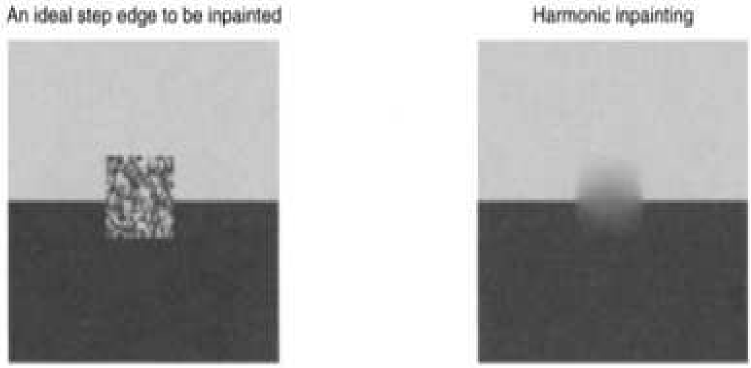


FIGURE 1. Harmonic inpainting (T. Chan and J. Shen [19]).

the image, spatially varying choices of the parameter $\alpha$ were proposed in the literature. We mention as an example the variant of the total variation (TV) functional, considered by D. Strong and T. Chan [36] which results in a multi-scale strategy with a uniform $\alpha$ updated at each scale [4]. Others strategies to choose such parameters are also developed within the statistical approach or using some a priori PDE [32] for the denoising problem. Note that the topological gradient method leads implicitly to such a choice by allowing the modification of the diffusion coefficients [5, 6].

We consider in this article a novel approach which consists of an adaptive method for the choice of such spatially varying regularization parameters. The method is well-suited for images with few textures and was successfully applied to the segmentation problems [8]. Loosely speaking, we start with a simple model (e.g. linear diffusion with a variable coefficient), then iteratively, an adaptive selection of the parameters based on some local information on the gradient magnitude is performed. The gradient information are available at the discrete level from the computed solution, thus the process is completely an a posteriori method without any reference to the continuous solution of (2). This amounts to change dynamically the reconstruction model in order to capture accurately the fine geometric structures