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PARALLEL PRECONDITIONERS FOR PLANE WAVE HELMHOLTZ AND MAXWELL SYSTEMS WITH LARGE WAVE NUMBERS

LONG YUAN, QIYA HU, AND HENGBIN AN

Abstract. A kind of non-overlapping domain decomposition preconditioner was proposed to solve the systems generated by the plane wave least-squares (PWLS) method for discretization of Helmholtz equation and Maxwell equations respectively in [13] and [14]. In this paper we introduce overlapping variants of this kind of preconditioner and give some comparison among these domain decomposition preconditioners. The main goal of this paper is to implement in parallel these domain decomposition preconditioners for the system with large wave numbers. The numerical results indicate that the preconditioners are highly scalable and are effective for solving Helmholtz equation and Maxwell's equations with large wave numbers.

Key words. Helmholtz equations, Maxwell's equations, large wave number, variational formulation, plane-wave basis, preconditioner, iteration counts.

1. Introduction

The plane-wave method differs from the traditional finite-element method (FEM) and the boundary-element method (BEM) in the sense that the basis functions are chosen as exact solutions of the governing differential equation without boundary conditions. This type of numerical method was first introduced to solve Helmholtz equations and was then extended to solve Maxwell's equations. Examples of this approach include the discontinuous enrichment method [1, 9], the variational theory of complex rays (VTCR) [21, 22], the ultra weak variational formulation (UWVF) [3, 4, 16], the plane-wave discontinuous Galerkin (PWDG) method [10, 12, 26], and the plane-wave least-squares (PWLS) method [20, 13, 14].

All methods described above fall into the class of Trefftz methods. An important advantage of the PWLS method over the others is that the stiffness matrix generated by the PWLS method is Hermitian positive definite, so it is easier to construct efficient preconditioners for this matrix. For example, a simple non-overlapping domain decomposition preconditioner for such stiffness matrix was constructed in [13] and [14]. The numerical results indicate that the system with middle wave numbers can be solved rapidly by the preconditioned CG method with the proposed preconditioner.

It is a difficult topic to construct an efficient preconditioner for Helmholtz equation or Maxwell's equations with large wave numbers. In fact, the existing domain decomposition methods (and multilevel methods) are inefficient to these equations except that the sizes of the coarse meshes are chosen as $O(1/\omega)$ (see, for example, [5, 8, 17, 25]), where ω denotes the wave number. It is clear that the restriction on the coarse mesh size is fatal in applications. Recently a kinds of successive preconditioners based on PML method are proposed to solve Helmholtz equations with large wave numbers (see [6, 7]). It has been shown that such preconditioners

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possess the optimal convergence independent of the mesh sizes [6]. It is certain that the results are most important advance in the solution method for Helmholtz equations with large wave numbers.

In this paper we are mainly interested in the parallel implementation of domain decomposition preconditioners for the systems generated by the PWLS method for discretization of Helmholtz equation and Maxwell equations with large wave numbers. Motivated by the non-overlapping domain decomposition preconditioner in [13] and [14], we construct overlapping domain decomposition preconditioner for such systems in the present paper. We give some comparison of iteration counts and computing times spent in PCG method with the non-overlaping preconditioner and the overlaping preconditioner. Numerical results indicate that the domain decomposition preconditioner with small overlap is more effective than the others when the wave number is large and the mesh size is small. In particular, we implement in parallel the domain decomposition preconditioner with one element overlap for solving the systems with large wave numbers, and we find that such preconditioner is strongly scalable and is very effective, without the restriction that the size of coarse meshes is $O(1/\omega)$.

The paper is organized as follows: In Section 2, we recall the proposed variational formulation for homogeneous Helmholtz equation and Maxwell's equations. In Section 3, we describe the plane wave discretization of the variational problem. In Section 4, we construct domain decomposition preconditioners for the stiffness matrix associated with the new variational problem. In Section 5, we address some key issues in the parallel implementation of the domain decomposition preconditioner in JASMIN framework. In Section 6, we report some numerical results to confirm the effectiveness of the new preconditioner for solving the system with large wave numbers.

2. Variational formulation for Helmholtz equation and Maxwell's equations

In this section we recall the Helmholtz equation and second-order system of Maxwell's equations.

The considered variational formulation is based on a triangulation of the solution domain. Suppose Ω is a bounded polyhedral domain in $\mathbb{R}^n$ $(n = 2, 3)$. Let Ω be divided into a partition in the sense that

$$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k, \quad \Omega_l \bigcap \Omega_j = \emptyset \quad \text{for } l \neq j.$$

Let $\mathcal{T}_h$ denote the triangulation comprising the elements $\{\Omega_k\}$, where h is the meshwidth of the triangulation. Define

$$\Gamma_{lj} = \partial\Omega_l \bigcap \partial\Omega_j \quad \text{for } l \neq j$$

and

$$\gamma_k = \overline{\Omega}_k \bigcap \partial\Omega \quad (k = 1, \ldots, N), \ \gamma = \bigcup_{k=1}^{N} \gamma_k.$$

2.1. The case of Helmholtz equation. Consider Helmholtz equations which is formalized, normalizing the wave's velocity to 1, by

$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in} \quad \Omega, \\ (\partial_{\mathbf{n}} + i\omega)u = g & \text{on} \quad \gamma. \end{cases} \tag{1}$$

The outer normal derivative is referred to by $\partial_{\mathbf{n}}$ and the angular frequency by ω.