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EQUIVALENCE BETWEEN RIEMANN-CHRISTOFFEL AND GAUSS-CODAZZI-MAINARDI CONDITIONS FOR A SHELL

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Abstract. We establish the equivalence between the vanishing three-dimensionnal Riemann-Christoffel curvature tensor of a shell and the two-dimensionnal Gauss-Codazzi-Mainardi compatibility conditions on its middle surface. Additionally we produce a new proof of Gauss theorema egregium and Bonnet theorem (reconstructing a surface from its two fundamental forms). This is performed in the very elegant framework of Cartan's moving frames.

Key words. Surfaces, 3D manifolds, Pfaffian systems, Frobenius integrability conditions, Riemann-Christoffel curvature tensor, moving frames, Cartan differential geometry, Tensorial calculus

1. Introduction

Let $\mathcal{D} \subset \mathbb{R}^3$ be a compact, connected, simply-connected manifold with boundary of class C^2. Let $X = (X^1, X^2, X^3)$ be a system of Cartesian coordinates and $x = (x^1, x^2, x^3)$ be a system of curvilinear coordinates in $\mathbb{R}^3$. The purpose of this paper is to revisit the integrability of the system of nonlinear partial differential equations (PDE)

$$\sum_{k,l=1,..3} \frac{\partial x^k}{\partial X^i} \delta_{kl} \frac{\partial x^l}{\partial X^j} = g_{ij}(X), \quad i, j = 1, 2, 3. \tag{1}$$

where g is a regular, twice covariant, positive definite bilinear form, for example of class $C^2(\mathcal{D})$. In some of their previous works Vallée and Fortuné have already addressed this question in the framework of Darboux's instantaneous rotation vectors [9, 10, 5]. We consider again this question by using Cartan setting as introduced in [2] . Let us note that the interest of our approach is that it does not rely on the knowledge of the radii of curvature nor on the principal directions of the shell as in [7].

The plan of this work is as follows: in the next section, for the sake of clarity we recall some definitions and properties satisfied by the metric, in section 3 we establish the Riemann-Christoffel compatibility conditions for a three-dimensional Riemannian manifold. In section 4 with the same Frobenius approach we establish the Gauss-Codazzi-Mainardi conditions for a surface embedded in $\mathbb{R}^3$. As a by-product, Weingarten's condition on the normal at each point of the surface is therefore recovered. In section 5 we address the equivalence of Riemann-Christoffel and Gauss-Codazzi-Mainardi compatibility conditions for a shell. Finally in section 6 we state Gauss Theorema egregium and Bonnet reconstruction theorem.

2. Notations, lemmas and assumptions

Let ε and δ be respectively the Levi-Civita symbol, and the Kronecker symbols. Einstein summation convention of repeated indices and exponents is applied. In

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$\mathbb{R}^n$, the identity matrix is denoted I_n. The transpose of the matrix R is the matrix R^t. The scalar product of two vectors u and v with components u^i and v^i is denoted

$$u \cdot v = \delta_{ij} u^i v^j, \ i,j = 1, ...n.$$

Let ω and λ be two 1-forms with components ω_k and λ_k. We define their tensor product as the covariant tensor $\omega \otimes \lambda$ with components:

$$(\omega \otimes \lambda)_{ij} = \omega_i \lambda_j, \ i,j = 1, ..., n.$$

Now let us enunciate two elementary results that we will use repeatedly.

Lemma 2.1 *(i) Let g be a given positive symmetric bilinear form in $\mathbb{R}^n \times \mathbb{R}^n$, there exists n independent 1-form ω^k (the volume n-form $\omega^1 \wedge \omega^2 \wedge ... \wedge \omega^n$ does not vanish) such that g can be expanded:*

$$g = \delta_{kl} \ \omega^k \otimes \omega^l. \tag{2}$$

(ii) The form g can be defined by it covariant (g_{ij}) and contravariant (g^{kl}) components related by

$$g_{ij} g^{jk} = \delta_i^k.$$

Let us denote by (e_k) the dual vector basis associated to (ω^k), i.e. $\omega^k(e_l) = \omega_i^k e_l^i = \delta_l^k$ and g^{-1} whose components are (g^{kl}) can be expanded:

$$g^{-1} = \delta^{ij} \ e_i \otimes e_j.$$

(iii) The expansion (2) is not unique. If there exists a second expansion such that $g(x) = \delta_{kl} \ \zeta^k \otimes \zeta^l$ and if the signs of the n-forms $\omega^1 \wedge \omega^2 ... \wedge \omega^n$ and $\zeta^1 \wedge \zeta^2 ... \wedge \zeta^n$ are the same, then there exists a rotation $R \in SO(n)$ such that:

$$\zeta^k = R_l^k \omega^l.$$

In the sequel Latin indices or exponents take their value in the set $\{1,2,3\}$, Greek indices or exponents take their value in the set $\{1,2\}$.

For $n = 3$ the exterior product of two 1-forms ω and λ is the 2-form $\omega \wedge \lambda$ with components

$$(\omega \wedge \lambda)_i = \delta_{ij} \ \varepsilon^{jkl} \ \omega_k \lambda_l, \ \ i,j,k,l = 1, ...3.$$

Let us consider two vectorial 1-forms $\omega = (\omega^1, \omega^2, \omega^3)$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ we use the compact expression $\omega \wedge \lambda$ to represent the vectorial 2-form with components $(\omega \wedge \lambda)^i = \delta^{ij} \ \varepsilon_{jkl} \ \omega^k \wedge \lambda^l, \ \ i,j,k,l = 1, ...3$. For scalar 1-forms ω, λ we remark that $\omega \wedge \lambda = -\lambda \wedge \omega$. However, for vectorial 1-forms $\omega = (\omega^1, \omega^2, \omega^3)$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ we remark that

$$\omega \wedge \lambda = \lambda \wedge \omega.$$

For example we have $(\omega \wedge \lambda)^1 = \omega^2 \wedge \lambda^3 - \omega^3 \wedge \lambda^2 = \lambda^2 \wedge \omega^3 - \lambda^3 \wedge \omega^2 = (\lambda \wedge \omega)^1$.

Lemma 2.2 *Let R be a rotation field. There exists a vectorial 1-form $\lambda = (\lambda^k)$ such that:*

$$dR = Rj(\lambda),$$

where d represents the exterior derivative and $j(\lambda) = \begin{pmatrix} 0 & -\lambda^3 & +\lambda^2 \\ +\lambda^3 & 0 & -\lambda^1 \\ -\lambda^2 & +\lambda^1 & 0 \end{pmatrix}$.

The proof is based upon the relationship $R^t R = I_3$ which implies that $R^t dR$ is antisymmetric. For all vectorial 1-forms λ, ω a direct computation yields:

$$j(\lambda) \wedge \omega = \lambda \wedge \omega.$$

Let us remark that $j(\lambda)$ is a vectorial 1-form for the Lie algebra $so(3)$ of $SO(3)$.