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BACKWARD EULER SCHEMES FOR THE KELVIN-VOIGT VISCOELASTIC FLUID FLOW MODEL

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Abstract. In this paper, we discuss the backward Euler method along with its linearized version for the Kelvin-Voigt viscoelastic fluid flow model with non zero forcing function, which is either independent of time or in $\mathbf{L}^\infty(\mathbf{L}^2)$. After deriving some bounds for the semidiscrete scheme, *a priori* estimates in Dirichlet norm for the fully discrete scheme are obtained, which are valid uniformly in time using a combination of discrete Gronwall's lemma and Stolz-Cesaro's classical result for sequences. Moreover, an existence of a discrete global attractor for the discrete problem is established. Further, optimal *a priori* error estimates are derived, whose bounds may depend exponentially in time. Under uniqueness condition, these estimates are shown to be uniform in time. Even when $\mathbf{f} = 0$, the present result improves upon earlier result of Bajpai *et al.* (IJNAM,10 (2013),pp.481-507) in the sense that error bounds in this article depend on $1/\sqrt{\kappa}$ as against $1/\kappa^r$, $r \geq 1$. Finally, numerical experiments are conducted which confirm our theoretical findings.

Key words. Viscoelastic fluids, Kelvin-Voigt model, a *priori* bounds, backward Euler method, discrete attractor, optimal error estimates, linearized backward Euler scheme, numerical experiments.

1. Introduction

Let Ω be a bounded convex polygonal or polyhedron domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$. Consider the following system of equations described by the Kelvin-Voigt viscoelastic fluid flow model (see, [20]): Find a pair $(\mathbf{u}, p)$ such that

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \kappa \Delta \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(x,t), \;\; x \in \Omega, \;\; t > 0 \tag{1}$$

with incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \;\; x \in \Omega, \, t > 0, \tag{2}$$

initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega, \; t \geq 0. \tag{3}$$

Here, $\mathbf{u} = \mathbf{u}(x,t)$ denotes the velocity vector, $p = p(x,t)$ is the pressure, $\nu > 0$ represents the kinematic coefficient of viscosity and κ is the retardation in time parameter. For some applications, we refer to [5],[6], [7] and references, therein.

Now, we quickly recall some theoretical developments on the Kelvin-Voigt model. Based on proof techniques of Ladyzenskaya [17] for establishing the wellposedness of the Navier Stokes system, Oskolkov [19, 20] has proved an existence of a global unique 'almost' classical solution in finite time interval for the initial and boundary value problem (1)-(3). Investigations on existence and uniqueness results for all time $t > 0$ have been further continued by him and his collaborators under various conditions on the forcing function $\mathbf{f}$, see [22] and [23].

For earlier results on numerical methods applied to the problem (1)-(3), we refer to [1] and [21]. Under the assumption that the solution is asymptotically stable

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as $t \to \infty$, Oskolkov [21] has proved convergence of spectral Galerkin approximations to the problem (1)-(3) in semi time axis $t \geq 0$. Later on, Pani *et al.* [26] have employed a variant of nonlinear semidiscrete spectral Galerkin method and derived optimal error estimates. Recently, Bajpai *et al.* [1] have applied finite element methods to discretize spatial variables and have established optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms and for the pressure term in $L^\infty(L^2)$- norm of the Kelvin-Voigt model with zero forcing function. It is, further, shown that both exact solution and semidiscrete solution decay exponentially in time. Moreover, the error estimates have similar exponential decay property. Subsequently, Bajpai *et al.* [2] have analyzed both first order backward Euler and second order backward difference schemes for the completely discretization of the problem (1)-(2), when the forcing function $\mathbf{f} = 0$. Firstly, an existence result is shown for the discrete nonlinear problem using a variant of Brouwer fixed point argument and optimal error estimates which reflect exponential decay property are proved. Note that their error bounds contain term like $\frac{1}{\kappa^r}$, where $r \geq 1$. For related articles on Navier-Stokes equations, see [11] and on Oldroyd model, refer to [9]-[10], [12], [24]-[27], [29]-[32].

When the non-zero forcing function $\mathbf{f} \in L^\infty(\mathbf{L}^2)$, which is crucial in the study of dynamical system, Pany *et al.* [27] have applied semidiscrete finite element method for the problem (1)-(3) and have proved the existence of a global attractor. New regularity results for the exact solution are established which are valid both uniformly in time as $t \mapsto \infty$ and in κ as $\kappa \mapsto 0$. With the help of Sobolev-Stokes projection introduced in [1], *a priori* optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms and for the pressure term in $L^\infty(L^2)$-norm are derived. Under uniqueness assumption, it is shown that error bounds are valid uniformly in time. When $\kappa = O(h^{2\delta})$, $\delta > 0$ small, where h is the spatial discretization parameter, it is, further, established that quasi-optimal error estimates are valid for small κ. Moreover, this articles concludes with several numerical experiments, which are based on backward Euler method with out error analysis. In continuation to the investigation in [27] on semidiscrete problem, in this article, a backward Euler method along with its linearized version for the time discretization is analyzed. *A priori* bounds for the discrete solution, specially in the Dirichlet norm are established using a combination of discrete Gronwall's lemma and Stolz-Cesaro theorem (see, pp 85-87 of [18]) for sequences, which can be thought of a discrete version of the L'Hospital's rule. It is, further, shown that the discrete problem has a global discrete attractor and then optimal error estimates are derived. More precisely, the following estimates are obtained

$$\|\mathbf{u}_h(t_n) - \mathbf{U}^n\| \leq Ck,$$

and

$$\|(p_h(t_n) - P^n)\| \leq \frac{C}{\sqrt{\kappa}}\, k,$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the backward Euler method and the pair $(\mathbf{u}_h(t_n), p_h(t_n))$ is the semi-discrete solution at time level t_n. Since constants in these error bounds depend on e^{Ct}, these results as in the Navier-Stokes case are valid locally. But under the uniqueness assumption, it is, further, shown that error estimates are valid uniformly in time. Then, using the contribution of semi-discrete error estimates from [27], we, finally, obtain error estimates for the complete discrete scheme.