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# A SIMPLE FINITE ELEMENT METHOD FOR NON-DIVERGENCE FORM ELLIPTIC EQUATIONS

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**Abstract.** We develop a simple finite element method for solving second order elliptic equations in non-divergence form by combining least squares concept with discontinuous approximations. This simple method has a symmetric and positive definite system and can be easily analyzed and implemented. Also general meshes with polytopal element and hanging node can be used in the method. We prove that our finite element solution approaches to the true solution when the mesh size approaches to zero. Numerical examples are tested that demonstrate the robustness and flexibility of the method.

**Key words.** Finite element methods, non-divergence form elliptic equations, polyhedral meshes.

## 1. Introduction

We consider a elliptic equations in non-divergence form

$$A : D^2u = f, \quad \text{in } \Omega, \tag{1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega$ is a convex polytopal domain in $\mathbb{R}^d$ with $d = 2, 3$. We assume that the model problem (1)-(2) has a unique solution and the coefficient tensor $A(x)$ is uniformly elliptic.

Non-divergence form elliptic partial differential equations have many applications in the areas such as stochastic processes and game theory [3]. In recent years, many numerical methods have been developed for second order elliptic equations in non-divergence form [1, 2, 4, 5, 6, 7] and the references therein.

The non-divergence nature of the problems makes it difficult to develop and analyze numerical algorithms for them since sophisticated Galerkin type numerical techniques cannot be applied directly. The goal of this work is to introduce a simple finite element method for non-divergence form partial differential equations which can be easily implemented and analyzed. This finite element method based on least squares methodology with discontinuous approximations has symmetric and positive definite system and is flexible to work with general meshes. We prove an optimal order error estimate for the finite element approximation in a discrete $H^2$ norm. However, our numerical results show optimal order of convergence in a discrete $H^1$ and $H^2$ norm.

## 2. Finite Element Methods

Let $\mathcal{T}_h$ be a partition of a domain $\Omega$ consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [8]. Denote

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by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by $h_T$ its diameter and mesh size $h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

We define a finite element space $V_h$ as follows for $k \geq 2$,

$$V_h = \{v \in L^2(\Omega) : \; v \in P_k(T), \; T \in \mathcal{T}_h\}. \tag{3}$$

Let elements $T_1$ and $T_2$ have $e$ as a common edge/face. We define a jump of $\phi$ on $e$ as

$$[\phi]_e = \begin{cases} \phi|_{\partial T_1} - \phi|_{\partial T_2}, & e \in \mathcal{E}_h^0, \\ \phi, & e \in \partial\Omega. \end{cases}$$

The order of $T_1$ and $T_2$ is non-essential as long as the difference is taken in a consistent way.

We introduce two bilinear forms as follows

$$\begin{aligned} s(v,w) &= \sum_{e\in\mathcal{E}_h} \int_e h_e^s [v][w] ds + \sum_{e\in\mathcal{E}_h^0} \int_e h_e^t [\nabla v] \cdot [\nabla w] ds, \\ a(v,w) &= \sum_{T\in\mathcal{T}_h} (A : D^2 v, \; A : D^2 w)_T + s(v,w), \end{aligned}$$

where $s$ and $t$ are two integers such that $s \geq -3$ and $t \geq -1$. For simplicity, we will let $s = t = -1$ in the rest of the paper.

**Algorithm 1.** *A numerical approximation for (1)-(2) can be obtained by seeking $u_h \in V_h$ satisfying the following equation:*

$$a(u_h, v) = (f, \; A : D^2 v) \quad \forall v \in V_h. \tag{4}$$

**Lemma 1.** *The finite element scheme (4) has a unique solution.*

*Proof.* It suffices to show that the solution of (4) is trivial if $f = 0$. Assuming $f = 0$ and taking $v = u_h$ in (4), we have

$$\sum_{T\in\mathcal{T}_h} (A : D^2 u_h, A : D^2 u_h)_T + s(u_h, u_h) = 0,$$

which implies that $A : D^2 u_h = 0$ on each element $T$ and $u_h \in C_0^1(\Omega)$. Thus $u_h$ is a solution of the problem (1)-(2) with $f = 0$. The uniqueness of the solution of the model problem (1)-(2) implies $u_h = 0$. $\square$

We define a semi-norm $|\!|\!| \cdot |\!|\!|$ as follows,

$$|\!|\!| v |\!|\!|^2 = a(v,v).$$

Similar to the proof of Lemma 1, we can prove that $|\!|\!| \cdot |\!|\!|$ define a norm in $V_h$.

## 3. Error Estimate

In this section, we will estimate the difference between the true solution $u$ and its finite element approximation $u_h$ from (4).

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [8] for details):

$$\|\varphi\|_e^2 \leq C \left( h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla\varphi\|_T^2 \right). \tag{5}$$