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SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHODS FOR LINEAR HYPERBOLIC EQUATIONS WITH SINGULAR INITIAL DATA

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Abstract. In this paper, we consider the discontinuous Galerkin (DG) methods to solve linear hyperbolic equations with singular initial data. With the help of weight functions, the superconvergence properties outside the pollution region will be investigated. We show that, by using piecewise polynomials of degree k and suitable initial discretizations, the DG solution is $(2k+1)$-th order accurate at the downwind points and $(k+2)$-th order accurate at all the other downwind-biased Radau points. Moreover, the derivative of error between the DG and exact solutions converges at a rate of $k+1$ at all the interior upwind-biased Radau points. Besides the above, the DG solution is also $(k+2)$-th order accurate towards a particular projection of the exact solution and the numerical cell averages are $(2k+1)$-th order accurate. Numerical experiments are presented to confirm the theoretical results.

Key words. Discontinuous Galerkin (DG) method, singular initial data, linear hyperbolic equations, superconvergence, weight function, weighted norms.

1. Introduction

In this paper, we apply discontinuous Galerkin (DG) methods to solve linear hyperbolic equation with non-smooth solution in one space dimension

$$u_t + u_x = 0, \quad (x,t) \in [0,2\pi] \times (0,T], \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in [0,2\pi], \tag{2}$$

where the initial solution $u_0(x)$ has a discontinuity at $x = c$, but is otherwise smooth. We consider problem with suitable Dirichlet boundary condition

$$u(0,t) = g(t) \tag{3}$$

such that the exact solution is smooth except along the characteristic line $x = t+c$. It is well known that the numerical solution has spurious oscillations around the discontinuity line, which is regarded as "pollution region". The early works studying error estimates of DG methods for hyperbolic problems with discontinuities were given by Johnson et. al. [16, 17, 18]. They proved that the width of the pollution region is of the size at most $\mathcal{O}(h^{\frac{1}{2}} \log(1/h))$ with linear space-time elements. Later, similar results were also obtained by Cockburn and Guzmán [10] and Zhang and Shu [26] with the RKDG methods. The main idea is to introduce special weight functions which are very small near the singularity and are close to 1 outside the pollution region. More recently, Yang and Shu [25] applied the same idea and proved the $(2k+1)$-th superconvergence in negative-order norms outside the pollution region. To our best knowledge of the authors, this is the only superconvergence result for DG methods applied to hyperbolic equations with singular exact solutions.

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The DG method was first introduced in 1973 by Reed and Hill [21], in the framework of neutron linear transport. Later, the method was applied by Johnson and Pitkäranta to a scalar linear hyperbolic equation and the L^p-norm error estimate was proved [17]. Subsequently, Cockburn et al. developed Runge-Kutta discontinuous Galerkin (RKDG) methods for hyperbolic conservation laws in a series of papers [13, 12, 11, 14]. Generally, we choose completely discontinuous piecewise polynomial space for DG methods. Hence, DG methods have several advantages such as high parallel efficiency, efficient h-p adaptivity, arbitrary order of accuracy and so on.

Superconvergence properties of DG methods for hyperbolic equations have been studied intensively, see [1, 2, 3, 4, 27, 20, 28, 8, 9, 24, 5, 6, 7] and the references therein. Many of the previous works are based on local error estimates or Fourier analysis, and the results only work for some special problems. In 2010, Cheng and Shu [9] applied energy analysis to obtain a $(k + 3/2)$-th superconvergence rate for the error between the DG solution and the particular projection of the exact solution. However, numerical experiments demonstrated the rate should be $k + 2$. Recently, Yang and Shu [24] extended the results in [9] to show that, with suitable initial discretization and upwind fluxes, the DG solution is $(k+2)$-th order superclose to the exact solution at the downwind-biased Radau points. The same convergence rate also works for the numerical cell averages. Subsequently, Cao et al. [5] proved a $(2k + 1)$-th order convergence rate of the error at the downwind point by constructing a special interpolation function. After that, in [7] and [6], the idea was applied to problems in two space dimensions and those in one space dimension with upwind-biased fluxes.

One of the most significant applications of the superconvergence is the construction of adaptive methods. The key point is to use the superconvergence properties to introduce a new numerical approximation which is superclose to the exact solution. Then the error between the two numerical approximations can be used as an error indicator to detect the regions with poor resolutions or singularities [19]. In this paper, we would like to analyze the error of the DG method for linear hyperbolic conservation law (1) outside the pollution region. The basic idea is to construct a suitable interpolation function u_I such that the DG solution u_h is $(2k + 1)$-th order accurate towards u_I under some weighted norms. By using the special properties of the weight functions we can prove several superconvergence results between the DG solution and the exact solution outside the pollution region. We will show that, under suitable initial discretizations, the DG solution is $(2k+1)$-th order accurate at the downwind points and $(k+2)$-th order accurate at all the other downwind-biased Radau points. Moreover, the derivative converges at a rate of $k + 1$ at all the interior upwind-biased Radau points. Besides the above, the DG solution is $(k + 2)$-th order accurate towards a particular projection of the exact solution and the numerical cell averages are $(2k + 1)$-th order accurate.

The organization of this paper is as follows. In Section 2, we will present preliminaries, including an introduction of DG scheme, some special projections, several elementary lemmas as well as the weight functions. In Section 3, we prove the main superconvergence results. Numerical experiments will be given in Section 4