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AN OPTIMAL *A POSTERIORI* ERROR ESTIMATES OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE SECOND-ORDER WAVE EQUATION IN ONE SPACE DIMENSION

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Abstract. In this paper, we provide the optimal convergence rate of *a posteriori* error estimates for the local discontinuous Galerkin (LDG) method for the second-order wave equation in one space dimension. One of the key ingredients in our analysis is the recent optimal superconvergence result in [W. Cao, D. Li and Z. Zhang, *Commun. Comput. Phys.* 21 (1) (2017) 211-236]. We first prove that the LDG solution and its spatial derivative, respectively, converge in the L^2-norm to $(p+1)$-degree right and left Radau interpolating polynomials under mesh refinement. The order of convergence is proved to be $p+2$, when piecewise polynomials of degree at most p are used. We use these results to show that the leading error terms on each element for the solution and its derivative are proportional to $(p+1)$-degree right and left Radau polynomials. These new results enable us to construct residual-based *a posteriori* error estimates of the spatial errors. We further prove that, for smooth solutions, these *a posteriori* LDG error estimates converge, at a fixed time, to the true spatial errors in the L^2-norm at $\mathcal{O}(h^{p+2})$ rate. Finally, we show that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h)$ rate. The current results improve upon our previously published work in which the order of convergence for the *a posteriori* error estimates and the global effectivity index are proved to be $p+3/2$ and $1/2$, respectively. Our proofs are valid for arbitrary regular meshes using P^p polynomials with $p \geq 1$. Several numerical experiments are performed to validate the theoretical results.

Key words. Local discontinuous Galerkin method, second-order wave equation, superconvergence, Radau points, *a posteriori* error estimation.

1. Introduction

In this paper, we analyze a residual-based *a posteriori* error estimates of the spatial errors for the semi-discrete local discontinuous Galerkin (LDG) method applied to the following one-dimensional linear wave equation

$$u_{tt} = u_{xx} + cu, \quad x \in [a,b],\ t \in [0,T], \tag{1a}$$

subject to the initial and periodic boundary conditions

$$u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad x \in [a,b], \tag{1b}$$

$$u(a,t) = u(b,t), \quad u_x(a,t) = u_x(b,t), \quad t \in [0,T], \tag{1c}$$

where c is assumed to be a constant. For the sake of simplicity, we only consider the case of periodic boundary conditions. However, this assumption is not essential. We note that if other boundary conditions (*e.g.*, Dirichlet or Neumann or mixed boundary conditions) are chosen, the LDG method can be easily designed; see [6, 10, 19, 39] for some discussion. In our analysis, the initial conditions are assumed to be sufficiently smooth functions so that the exact solution, $u(x,t)$, is a smooth function on $[a,b] \times [0,T]$.

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The discontinuous Galerkin (DG) method was first developed in the early 1970s by Reed and Hill [34] for solving hyperbolic conservation laws containing only first order spatial derivatives. However, in the last two decades it has become attractive as a powerful simulation tool for solving many partial differential equations. The DG method is a class of finite element methods, using discontinuous, piecewise polynomials as the numerical solution and the test functions. The DG method combines the best proprieties of the classical continuous finite element and finite volume methods such as consistency, flexibility, stability, conservation of local physical quantities, robustness and compactness. Recently, DG methods become highly attractive and popular, mainly because these methods are high-order accurate, nonlinear stable, highly parallelizable, easy to handle complicated geometries and boundary conditions, and capable to capture discontinuities without spurious oscillations. Since then the DG method has been analyzed and extended to a wide range of applications. In particular, for time dependent problems, Cockburn and Shu [26] extended the DG method to solve first-order hyperbolic partial differential equations of conservation laws. They used a method of lines which consists of applying the DG scheme to approximate the problem in space and then to apply a Runge-Kutta scheme in time to obtain an RKDG scheme. They further developed the local DG (LDG) method for convection-diffusion problems [27]. The proceeding of Shu [36] contain a more complete and current survey of the DG method and its applications.

The LDG method we discuss in this paper is an extension of the DG method aimed at solving differential equations containing higher than first-order spatial derivatives. The LDG method for solving convection-diffusion problems was first introduced by Cockburn and Shu in [27]. LDG methods are robust and high-order accurate, can achieve stability without slope limiters, and are locally (elementwise) mass-conservative. This last property is very useful in the area of computational fluid dynamics, especially in situations where there are shocks, steep gradients or boundary layers. Moreover, LDG methods are extremely flexible in the mesh-design; they can easily handle meshes with hanging nodes, elements of various types and shapes, and local spaces of different orders. They further exhibit strong superconvergence that can be used to estimate the discretization errors. LDG schemes have been successfully applied to hyperbolic, elliptic, and parabolic partial differential equations [6, 26, 28, 29, 19, 38, 33, 27, 15, 18, 19, 7, 2, 17, 3, 4], to mention a few. A review of the LDG methods is given in [7, 11, 17, 25, 23, 16, 24, 19, 37, 39, 14].

In [6], we investigated the superconvergence properties of the LDG method for the second-order wave equation in one space dimension. We performed an error analysis on one element and showed that the p-degree LDG solution and its spatial derivative are $\mathcal{O}(h^{p+2})$ superconvergent at the roots of $(p+1)$-degree right and left Radau polynomials, respectively. Computational results showed that global superconvergence holds for LDG solutions. We used these results to construct asymptotically correct *a posteriori* error estimates by solving local steady problem with no boundary conditions on each element. However, we only presented several numerical results suggesting that the global spatial error estimates converge to the true errors under mesh refinement where temporal errors are assumed to be negligible. In [10], we analyzed the LDG method introduced by the author in [6] for solving the one-dimensional second-order wave equation. We used a suitable projection of the initial conditions for the numerical scheme and proved optimal L^2