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FINITE ELEMENT METHOD TO CONTROL THE DOMAIN SINGULARITIES OF POISSON EQUATION USING THE STRESS INTENSITY FACTOR : MIXED BOUNDARY CONDITION

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Abstract. In this article, we consider the Poisson equation on a polygonal domain with the domain singularity raised from the changed boundary conditions with the inner angle $\omega > \frac{\pi}{2}$. The solution of the Poisson equation with such singularity has a singular decomposition: regular part plus singular part. The singular part is a linear combination of one or two singular functions. The coefficients of the singular functions are usually called stress intensity factors and can be computed by the extraction formula. In [11] we introduced a new partial differential equation which has 'zero' stress intensity factor using this stress intensity factor, from whose solution we can obtain a very accurate solution of the original problem simply by adding singular part. Although the method in [11] works well for the Poisson problem with Dirichlet boundary condition, it does not give optimal results for the case with stronger singularity, for example, mixed boundary condition with bigger inner angle. In this paper we give a revised algorithm which gives optimal convergences for both cases.

Key words. Finite element, singular function, dual singular function, stress intensity factor.

1. Introduction

Let Ω be an open, bounded polygonal domain in $\mathbb{R}^2$ and let Γ_D and Γ_N be a partition of the boundary of Ω such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. As a model problem, we consider the following Poisson equation with mixed boundary conditions:

$$(1) \qquad \begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_N, \end{cases}$$

where $f \in L^2(\Omega)$, Δ stands for the Laplacian operator and ν denotes the outward unit vector normal to the boundary.

If $\Gamma_N = \emptyset$ and the domain is convex or smooth, we expect to have an optimal convergence rate with standard finite element method. But it is not true for Poisson problems defined on non-convex domains. The solution u of the Poisson equation with such singularity has a singular decomposition: $u = w + \lambda\eta s$, where $w \in H^2(\Omega) \cap H^1_D(\Omega)$, η is a smooth cut-off function and s is a singular function. Here, $H^1_D(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$.

Such lack of regularity affects the accuracy of the finite element approximation. There were several approaches in the literatures for overcoming this difficulty. One is based on local mesh refinement (see, e.g., [1, 13, 14, 15, 16] and references therein). The advantage of the method of local mesh refinement is that the knowledge of the exact forms of the singular functions is not needed. Another is done by augmenting the space of trial functions in which one looks for the approximate solution (see, e.g.,[2, 3, 4, 6, 7] and references therein). In [4], they introduced a new approach

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that pose a partial differential equation with regular part of the solution, then compute the stress intensity factor and the solution.

The coefficient λ of the singular function is usually called stress intensity factor and can be computed by the extraction formula. In [11] we introduced a new partial differential equation which has ‘zero’ stress intensity factor, from whose solution we got accurate solution of the original problem simply by adding singular part.

Although the method in [11] works well for the Poisson problem with Dirichlet boundary condition, it does not give optimal results for the case with stronger singularity, for example, mixed boundary condition with bigger inner angle. In this paper we give a revised algorithm which gives an optimal convergence for both cases. In this paper we assume there is only one singular point raised from mixed Dirichlet and Neumann boundary condition.

We will use the standard notations and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$, and the associated inner products are denoted by $(\cdot,\cdot)_{t,\Omega}$, with their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively. However, we will omit Ω if there is no chance of misunderstanding.

In section 2 we give the forms of singular functions and the dual singular functions together with the extraction formula. In section 3 we suggest a revised algorithm and some theorems. In section 4 and 5 we give finite element approximation and some examples with numerical results.

2. Singular functions and extraction formula

We use a cut-off function to isolate the singular behavior of the problem. So, we first give the definition of the cut-off functions with parameters. Set

$$B(r_1;r_2) = \{(r,\theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega \quad \text{and} \quad B(r_1) = B(0;r_1).$$

We define a family of cut-off functions of r as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B(\tfrac{1}{2}\rho), \\ \tfrac{15}{16}\left\{\tfrac{8}{15} - p(r) + \tfrac{2}{3}p(r)^3 - \tfrac{1}{5}p(r)^5\right\} & \text{in } \bar{B}(\tfrac{1}{2}\rho;\,\rho), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho), \end{cases} \tag{2}$$

where $p(r) = 4r/\rho - 3$. Here, ρ is a parameter which will be determined so that the singular part $\eta_\rho s$ has the same boundary condition as the solution u of the Model problem, s is the singular function which is given in (3)–(7). Note that $\eta_\rho(r)$ is C^2.

If $\Gamma_N = \emptyset$ with the inner angle $\omega, \pi < \omega < 2\pi$, as in [11], we have one singular function and its dual singular function:

$$s = s(r,\theta) = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega} \quad \text{and} \quad s_- = s_-(r,\theta) = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}. \tag{3}$$

If $\Gamma_N \neq \emptyset$ and the boundary condition changes its type at vertices with the inner angle ω, $\frac{\pi}{2} < \omega < 2\pi$, we have a list of the singular functions:

1) **D/N** If $\frac{\pi}{2} < \omega \leq \frac{3\pi}{2}$, there is a singular function of the form

$$s_1 = s_1(r,\theta) = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}; \tag{4}$$

If $\frac{3\pi}{2} < \omega < 2\pi$, there are two singular functions of the form

$$s_1 = s_1(r,\theta) = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega} \quad \text{and} \quad s_3 = s_3(r,\theta) = r^{\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}; \tag{5}$$