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A THIRD ORDER LINEARIZED BDF SCHEME FOR MAXWELL'S EQUATIONS WITH NONLINEAR CONDUCTIVITY USING FINITE ELEMENT METHOD

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Abstract. In this paper, we study a third order accurate linearized backward differential formula (BDF) type scheme for the nonlinear Maxwell's equations, using the Nédelec finite element approximation in space. A purely explicit treatment of the nonlinear term greatly simplifies the computational effort, since we only need to solve a constant-coefficient linear system at each time step. An optimal L^2 error estimate is presented, via a linearized stability analysis for the numerical error function, under a condition for the time step, $\tau \leq C_0^* h^2$ for a fixed constant C_0^*. Numerical results are provided to confirm our theoretical analysis and demonstrate the high order accuracy and stability (convergence) of the linearized BDF finite element method.

Key words. Maxwell's equations with nonlinear conductivity, convergence analysis and optimal error estimate, linearized stability analysis, the third order BDF scheme.

1. Introduction

This paper is concerned with the nonlinear Maxwell's equations

$$\epsilon \mathbf{E}_t + \sigma(\mathbf{x}, |\mathbf{E}|)\mathbf{E} - \nabla \times \mathbf{H} = 0, \ in\ \Omega \times (0, +\infty), \tag{1}$$

$$\mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0, \ in\ \Omega \times (0, +\infty), \tag{2}$$

with initial and boundary conditions

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(x), \ \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(x), \ in\ \Omega, \tag{3}$$

$$\mathbf{n} \times \mathbf{E} = 0 \ \ and \ \ \mathbf{n} \cdot \mathbf{H} = 0, \ on\ \Gamma \times (0, +\infty), \tag{4}$$

where Ω is a bounded, convex, simply-connected domain in R^3 with a regular boundary $\Gamma = \partial\Omega$, $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{H}(\mathbf{x}, t)$ represent the electric and magnetic fields, $\mathbf{n}$ is the outward normal vector on Γ, and the positive constants ϵ and μ stand for the permittivity and the magnetic permeability, respectively. In addition, $\sigma = \sigma(\mathbf{x}, s)$ is a real valued function representing the electric conductivity.

The system $(1)-(4)$ have been investigated in [6, 31, 32]. The authors proved the existence of the weak solution for a nonlinear function $\mathbf{J}(\mathbf{E}) = \sigma(|\mathbf{E}|)\mathbf{E}$, with $\sigma(s)$ monotonically increasing. In [4], the authors presented the existence and uniqueness of the scheme by discretizing the time domain and taking the limit for infinitely small time-step. In section 5 of [4], the authors also proved that the solutions converges to the quasi-state state (no time derivative for equation involving $\mathbf{E}$) when the permittivity $\epsilon \to 0$. Also when ϵ goes zero, numerical example indicated that numerical scheme converges to the quasi static state as part of verification. In [5], the authors proved existence and uniqueness of the discrete fields using the monotone operator theory (see [27]). In [26], the authors studied a time dependent eddy current equation, established the existence and uniqueness of a weak solution in suitable function space, designed a nonlinear time discrete approximation scheme

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based on the Rothe's method and proved the convergence of approximation to a weak solution.

Numerical analysis for the nonlinear system have also been extensively carried out, see [2, 3, 4, 5, 9, 10, 11, 12, 13, 14, 18, 22, 23, 25, 26]. In addition, nonlinear schemes have been proposed and analyzed in many literatures. In [4], the authors presented a numerical scheme to solve coupled Maxwell's equations with a nonlinear conductivity, with the backward Euler discretization in time and mixed conforming finite elements in space. And also, a mixed finite element method for the Maxwell's equations with a nonlinear boundary condition was studied in [25]. In [10], the authors proposed a fully-discrete finite element method to solve the time-domain metamaterial Maxwell's equations, which can be reduced to a vector wave integro-differential equation involving just one unknown. Some related works can also be found in [9, 11, 12, 18]. In [5], the authors proposed a numerical scheme based on backward Euler discretization in time and curl-conforming finite element in space to solve Maxwell's equations with nonlinear conductivity in the form of a power law. As a result, its convergence was proved, based on the boundedness of the second derivative in the dual space by the Minty-Browder technique. In [3], the authors developed a fully-discrete $(T, \psi) - \psi_e$ finite element decoupled scheme to solve time-dependent eddy current problem with multiple-connected conductors. Subsequently, an improved convergence rate analysis was provided in [14]. A few more earlier works are also available in [23, 26].

Clearly, linearized schemes are much more efficient than nonlinear schemes for solving nonlinear equations, since only one linear system solver is needed in the former one, while the latter one always requires a nonlinear iteration solver at each time step. For example, a new approach was developed in [16], based on a temporal-spatial error splitting technique by introducing a corresponding time-discrete system. Similarly, a linearized backward differential formula (BDF) type scheme was applied to the time-dependent nonlinear thermistor equation in [7]. The linearized backward Euler scheme for the nonlinear Joule heating equation was studied in [8]. In [29], the authors presented an optimal L^2 error estimate of a linearized Crank-Nicolson Galerkin FEM for a generalized nonlinear Schrödinger equation, without any time step size restriction.

In turn, an important question arises: Is it possible to design higher order (≥ 3) linearized temporal discretization for the nonlinear Maxwell's equations, with a convergence analysis available? In this paper, we give an affirmative answer to this question. We propose a third order accurate, linearized BDF type FEM method for the Maxwell's equations and provide a theoretical error analysis for the proposed scheme. It is well-known that the 3rd order BDF temporal scheme is not A-stable; in fact, its stability domain does not contain any part on the purely imaginary axis, where all the eigenvalues of the linear Maxwell operator are located. This fact makes a theoretical analysis for the 3rd order BDF method applied to the Maxwell equation highly challenging. To overcome this subtle difficulty, we take an inner product with the numerical error equation by $e^{n+1} + (\lambda_0 + 1)(e^{n+1} - e^n)$ (with $\lambda_0 > 0$ and e^k denoted as the numerical error function at time step t^k), and employ a telescope formula established in [20]. Moreover, due to the hyperbolic nature of the Maxwell's equation, a requirement for the time step size, $\tau \leq C_0^* h^2$ (with C_0^* a fixed constant), has to be imposed to pass through the numerical error estimate. Another key technical contribution in the convergence analysis is to obtain an L^∞ bound of the numerical solution, via a linearized stability analysis for the numerical error function; by contrast, such an L^∞ bound was not needed in the previous