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A POSTERIORI ERROR ESTIMATES FOR MIXED FINITE ELEMENT GALERKIN APPROXIMATIONS TO SECOND ORDER LINEAR HYPERBOLIC EQUATIONS

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Abstract. In this article, *a posteriori* error analysis for mixed finite element Galerkin approximations of second order linear hyperbolic equations is discussed. Based on mixed elliptic reconstructions and an integration tool, which is a variation of Baker's technique introduced earlier by G. Baker (SIAM J. Numer. Anal., 13 (1976), 564-576) in the context of *a priori* estimates for a second order wave equation, *a posteriori* error estimates of the displacement in $L^\infty(L^2)$-norm for the semidiscrete scheme are derived. Finally, a first order implicit-in-time discrete scheme is analyzed and *a posteriori* error estimators are established.

Key words. Second order linear wave equation, mixed finite element methods, mixed elliptic reconstructions, semidiscrete method, first order implicit completely discrete scheme, and *a posteriori* error estimates.

1. Introduction

In this paper, we discuss *a posteriori* error estimates for mixed finite element Galerkin approximations to the following class of second order linear hyperbolic problems:

$$u_{tt} - \nabla \cdot (A\nabla u) = f \qquad \text{in } \Omega \times (0,T], \tag{1}$$

$$u|_{\partial\Omega} = 0 \qquad u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1. \tag{2}$$

Here, $\Omega \subset I\!R^2$ is a bounded polygonal domain with boundary $\partial\Omega$, $0 < T < \infty$, $u_t = \frac{\partial u}{\partial t}$ and $A(x) = (a_{ij}(x))_{1\le i,j\le 2}$ is a symmetric and uniformly positive definite matrix. All the coefficients a_{ij}'s are smooth functions of x with uniformly bounded derivatives in $\bar{\Omega}$. Moreover, the initial functions $u_0 = u_0(x)$, $u_1 = u_1(x)$ and the forcing function $f = f(x,t)$ are assumed to be smooth functions in their respective domains.

In recent years, there has been a growing demand for designing reliable and efficient space-time algorithms for the numerical computation of time dependent partial differential equations. Most of these algorithms are based on *a posteriori* error estimators, which provide appropriate tools for adaptive mesh refinements. For elliptic boundary value problems, *a posteriori* error estimates are well developed (see, [3, 32]). Adaptivity with *a posteriori* error control for parabolic problems has also been an active research area for the last two decades (cf. [18, 33, 25, 30, 8, 9, 5] and references, therein). For the time discretization, some results on *a posteriori* error estimations for abstract first order evolution problems are available in the literature (cf. [4, 21, 26, 28, 30]).

In the context of second order wave equations, only few results are available on *a posteriori* error analysis, see, [24, 1, 14, 13, 7, 31]. Further, it is observed that the design and implementation of adaptive algorithms for these equations based on rigorous *a posteriori* error estimators are less complete compared to elliptic

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and parabolic equations. Based on a space-time finite element discretization with basis functions being continuous in space and discontinuous in time, *a priori* and *a posteriori* error estimates for second order linear wave equations are proved in [24]. Asymptotically exact *a posteriori* estimates for the standard finite element method are proposed and analyzed in [1, 2] by solving a set of local elliptic problems. The recent results in [7, 20] cover only first order time discrete schemes. In [7], the second order wave equation is written as a first order system and a first order implicit backward Euler scheme in time is used with continuous piecewise affine finite elements in space. Further, rigorous *a posteriori* bounds have been established using energy arguments and adaptive algorithms based on the *a posteriori* bounds are discussed. In [20], based on Baker's technique *a posteriori* bounds are derived for the semidiscrete scheme in $L^\infty(L^2)$-norm and for first order implicit-in-time fully discrete schemes in $\ell^\infty(L^2)$-norm. The fully discrete analysis relies crucially on a novel time reconstruction satisfying a local vanishing-moment property, and on a space reconstruction technique used earlier in [28] for parabolic problems. In [14], an adaptive algorithm in space and time which is based on Galerkin space-time discretizations leading to Newmark scheme is analyzed. Further, goal oriented *a posteriori* error estimates are derived and some numerical results are provided to demonstrate the efficiency of error estimators. In [31], the author has studied an anisotropic *a posteriori* error estimate for a finite element discretization of a two dimensional wave equation. The estimate is derived in the $L^2(0, T, H^1(\Omega))$-norm and it turns out to be sharp on anisotropic meshes.

For higher order time reconstruction for abstract second order evolution equations, one may refer to the recent papers [23, 22]. In [23], an adaptive time stepping Galerkin method is analyzed for second order evolution problems. Based on the energy approach and the duality argument, optimal order *a posteriori* error estimates and *a posteriori* nodal superconvergence results have been derived. An adaptive time stepping strategy is discussed and some numerical experiments are conducted to assess the effectiveness of the proposed scheme. In a recent work [22], second order explicit and implicit two-step time discretization schemes such as leap-frog and cosine methods are discussed and *a posteriori* estimates using a novel time reconstruction are derived. Further, some numerical experiments are conducted to confirm their theoretical findings.

For space-time adaptivity, the finite element discretization depends on the space-time variational formulation and its error indicators include both space and time errors. Recently, attempts have been made to exploit elliptic reconstruction to prove optimal *a posteriori* error estimates in finite element methods for parabolic problems [28]. In fact, the role of the elliptic reconstruction operator in *a posteriori* estimates is quite similar to the role played by elliptic projection introduced earlier by Wheeler [34] for recovering optimal *a priori* error estimates of finite element Galerkin approximations to parabolic problems. This analysis is, further, developed for completely discrete scheme based on backward Euler method [26], for maximum norm estimates [17] and for discontinuous Galerkin methods for parabolic problems [21]. In recent works [29] and [27], the analysis is further extended to mixed FE Galerkin methods applied to parabolic problems.

In this article, an *a posteriori* analysis is discussed for mixed finite element Galerkin approximations of a class of second order linear hyperbolic problems. One notable advantage of mixed finite element scheme is that it offers a simultaneous approximations of displacements and stresses, resulting in better convergences rates for the stress variable. This property is important in applications such as