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A SIMPLE FINITE ELEMENT METHOD OF THE CAUCHY PROBLEM FOR POISSON EQUATION

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Abstract. In this paper, we introduce a simple method for the Cauchy problem. This new finite element method is based on least squares methodology with discontinuous approximations which can be implemented and analyzed easily. This discontinuous Galerkin finite element method is flexible to work with general unstructured meshes. Error estimates of the finite element solution are derived. The numerical examples are presented to demonstrate the robustness and flexibility of the proposed method.

Key words. Finite element methods, Cauchy problem, polyhedral meshes.

1. Introduction

We consider the Cauchy problem for Poisson equation

$$\begin{aligned} \Delta u =& f, \quad &\text{in} \quad \Omega, \\ u =& 0, \quad &\text{on} \quad \Gamma_1, \\ \nabla u \cdot \boldsymbol{n} =& g, \quad &\text{on} \quad \Gamma_1, \end{aligned} \tag{1}$$

where Ω is a bounded convex polytopal domain in $\mathbb{R}^d$ with $d = 2, 3$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Assume that Γ_1 is simply connected.

The Cauchy problem (1) is well-known to be ill-posed [1, 4]. It has applications in many different areas such as plasma physic, electrocardiography, and corrosion non-destructive evaluation (e.g., [5, 12, 16, 21]). Due to the ill-posedness, the numerical approximation of the Cauchy problem is very difficult and challenging. Traditionally, regularization techniques, such as Tikhonov regularization [26] and the quasi-reversibility approach [23], were used to provide robust numerical schemes. Many different finite element methods have also been developed for solving the Cauchy problem (1). In [15, 24, 25], Galerkin type approaches are proposed based on structured grids or special formulation of the continuous problem. The regularization techniques are also used in finite element settings, e.g., [6, 3, 7, 13]. In [2, 11, 19, 18], the Cauchy problem (1) is reformulated as minimization problems and then solved numerically with possible regularizations. More recently, primal-dual formulation is proposed and solved by discontinuous Galerkin (DG) finite element methods with suitable stabilization/regularization, see [9, 10].

The purpose of this paper is to develop a simple finite element method to approximate the solution of the Cauchy problem (1) when it exists and is unique. This method is designed aiming on easy implementation and easy error analysis. The methodology of the scheme is combining the least squares technique with discontinuous approximations. Suitable stabilization terms are added to ensure the

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stability of the discretization. As a result, our method leads to a symmetric and positive definite linear system of equations and is flexible to use on general polygonal meshes with hanging nodes. We prove that our discontinues finite element solution approaches to the solution of the model problem (1) when the mesh size approaches to zero. Convergence rate are studied in both energy norm and L^2-norm based on the conditional stability of the continues Cauchy problem. Comparing with existing methods, our approach is attractive due to its simplicity. The numerical results also show the efficiency of the proposed approach which confirms our theoretical results.

The rest of the paper is organized as follows. In Section 2, we recall Cauchy problem and its conditional stability results based on a traditional weak formulation. Our new simple discretization is given in Section 3. We study its stability and error estimates in Section 4 and 5, respectively. Finally, we present some numerical experiments to demonstrate the stability of the WG formulation in Section 6 .

2. Cauchy Problem

We denote the standard Lebesgue sapces by $L^2(\mathcal{D})$ and $\mathcal{D} \in \mathbb{R}^d$, $d = 2, 3$, with corresponding norms $\|\cdot\|_{L^2(\mathcal{D})}$ (or $\|\cdot\|_{\mathcal{D}}$). $H^s(\mathcal{D})$ denote the standard Sobolev space of index $s \geq 0$ along with the corresponding norm and semi-norm $\|\cdot\|_{H^s(\mathcal{D})}$ (or $\|\cdot\|_{s,\mathcal{D}}$) and $|\cdot|_{H^s(\mathcal{D})}$ (or $|\cdot|_{s,\mathcal{D}}$), respectively.

For the Cauchy problem (1), if the $(d-1)$-measure of Γ_2 is nonempty, it is an ill-conditioned problem. In practice, as shown in [4], such Cauchy problem is not well-posed due to measurement errors. However, following the traditional arguments, if the underlying physical process is stable, i.e., if the boundary data are known on the whole boundary, then the problem is well-posed, it is natural to assume that the Cauchy problem (1) has a unique solution in the idealized case with unperturbed data. Therefore, we assume that $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\Gamma_1)$, and that there is a unique solution $u \in H^2(\Omega)$ satisfies (1). Our analysis will be based on this assumption and the so-called conditional stability described later in Section 2.2.

2.1. A Traditional Weak Formulation. In order to introduce the conditional stability of the Cauchy problem (1), we need to first look at the weak formulation of the Cauchy problem (1). Following [1], we introduce two Sobolev spaces

$$H^1_{\Gamma_1}(\Omega) := \{v \in H^1(\Omega) : \ v|_{\Gamma_1} = 0\},$$

and

$$H^1_{\Gamma_2}(\Omega) := \{v \in H^1(\Omega) : \ v|_{\partial\Omega\setminus\Gamma_1} = 0\}.$$

The weak formulation for (1) is: find $u \in H^1_{\Gamma_1}(\Omega)$ such that

$$a_0(u, v) = l(v), \quad \forall\ v \in H^1_{\Gamma_2}(\Omega), \tag{2}$$

where

$$a_0(u, v) := \int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x,$$

and

$$l(v) := -\int_\Omega f v \mathrm{d}x + \int_{\Gamma_1} g v \mathrm{d}s.$$

Again, we are not assuming this weak formulation of Cauchy problem is well-posed since inf-sup stability does not hold in general [4].