INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 4-5, Pages 627–645

STOCHASTIC SPLINE-COLLOCATION METHOD FOR CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY RANDOM ELLIPTIC PDE

BENXUE GONG, LIANG GE, TONGJUN SUN, WANFANG SHEN, AND W.B. LIU

Abstract. In this paper, we investigate a stochastic spline-collocation approximation scheme for an optimal control problem governed by an elliptic PDE with random field coefficients. We obtain the necessary and sufficient optimality conditions for the optimal control problem and establish a scheme to approximate the optimality system through the discretization with respect to the spatial space by finite elements method and the probability space by stochastic spline-collocation method. We further investigate Smolyak approximation schemes, which are effective collocation strategies for smooth problems that depend on a moderately large number of random variables. For more general control problems where the state may be non-smooth with respect to the random variables in some areas, we adopt a domain decomposition strategy to partition the random space into smooth and non-smooth parts and then apply Smolyak scheme and spline approximation respectively. A priori error estimates are derived for the state, the co-state and the control variables. Numerical examples are presented to illustrate our theoretical results.

Key words. Random elliptic PDE, priori error estimates, stochastic spline-collocation method, Smolyak approximation, optimal control problem, deterministic constrained control.

1. Introduction

In recent years, there are increasing interests in modeling uncertainty in many complex physical and engineering systems, such as uncertain parameters, coefficients, forcing term, and boundary conditions. It is well known that these systems can be described by stochastic partial differential equations(SPDEs). Since stochastic PDEs are conveniently used in many areas, such as fluid flows in porous media,chemistry, transport of pollutants in groundwater and oil recovery processes, the numerical solutions for Stochastic PDEs have been a main subject of growing interest in the scientific community([4]-[22]).

The well-known Monte Carlo (MC) method is the most commonly used method for simulating stochastic PDEs and for dealing with the statistic characteristics of the solution [4, 5]. Although MC method only needs to do repetitive deterministic simulations, it is a rather computationally expensive method for the reason that the statistic convergence rate is relatively slow, especially when there are large amounts of computations in the deterministic systems. Another alternative to the Monte Carlo method is the so-called stochastic Galerkin method[9, 15] for solving stochastic PDEs with random fields input data. This method allows us to utilize standard approximations in space(finite elements, finite volumes, spectral or h-p finite elements, etc.) and polynomial approximation in the probability domain, either on full polynomial spaces [16, 20, 21], tensor product polynomial spaces [17, 18, 19], or on piecewise polynomial spaces [6, 7, 8, 17]. By applying stochastic Galerkin method, we can utilize the regularity of the solution and acquire faster convergence rates. However, in general, this technique requires to solve a system

Received by the editors on December 1, 2016, accepted on March 26, 2017.

2000 *Mathematics Subject Classification.* 65M55, 65N30.

of equations that couples all degrees of freedom when approximating the stochastic systems.

Due to this issue, the stochastic collocation method has gained much attention recently in the computational community [10, 11, 12, 13, 20], which was originally introduced in [10,20]. In principle, stochastic collocation method consists of a Galerkin approximation in physical space and a collocation in the zeros of suitable tensor product orthogonal polynomials (Gauss points) in the probability space[10]. Compared with stochastic Galerkin methods, this method solves uncoupled deterministic PDEs at the collocation points that are trivially parallelizable, as in the Monte Carlo method. This method also can treat efficiently the case of dependent random variables by introducing an auxiliary density $\hat{\rho}$. And this method deals easily with unbounded random variables such as Gaussian or exponential variables. Hence, stochastic collocation is an attractive method for computing solutions of stochastic PDEs with random field input data.

In many applications, optimization of physical and engineering systems can be formulated as optimal control problems that are constrained by PDEs. Computational methods for deterministic optimal control problems constrained by PDEs have been well developed and investigated for several decades([1]-[3],[23]-[29]). Recently efficient numerical methods for optimal control problem governed by stochastic PDEs are becoming a new hot topic. Comparison with the deterministic optimal control, efficient computation of stochastic optimal control problems constrained by stochastic PDEs is still in its infancy, see the very recent work([30]-[37]). Based on the work([6]-[22]), [30] dealed with optimal control governed by random steady PDEs with deterministic Neumann boundary control, and the existence of an optimal solution and of a Lagrange multiplier were demonstrated. The authors also proposed the stochastic finite element solution of the optimality system and estimated its error through the discretizations with respect to both spatial and random parameter spaces. In [31], one-shot stochastic finite element methods were used to find approximate solutions with 'pure' stochastic control function as well as 'semi' stochastic control function for an optimal control problem constrained by stochastic steady diffusion problems. In [32] and [33], stochastic optimal control governed by stochastic elliptic PDEs with deterministic distributed control function were introduced, and the authors proved the existence of the optimal solution, established the validity of the Lagrange multiplier rule and obtained stochastic optimality system. Computationally, the numerical solutions of the optimality system were given by the stochastic finite element method. In [34], the author proposed framework combines space-time multigrid methods with sparse-grid collocation techniques to solve nonlinear parabolic optimal control problems with random coefficients for unconstrained control. In [35], we studied an optimal control problem governed by an elliptic PDE with random field in coefficients and constrained control, and obtained the necessary and sufficient optimality conditions by applying the well-known Lions' lemma. Then a stochastic finite element approximation scheme is applied and the a priori error estimate for the state, the co-state and the control variables is derived. In [36] a stochastic finite element approximation scheme and the a priori error estimate for the state, the co-state and the control variables were developed for an optimal control problem governed by an elliptic integro-differential equation with random coefficients. Furthermore in [37], stochastic finite element is applied to an optimal control problem governed by a parabolic PDE with random field in its coefficients, and a priori error estimates for the state, the co-state and the control variables have been given. However, to our best knowledge, there has been