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A FINITE ELEMENT METHOD FOR THE ONE-DIMENSIONAL PRESCRIBED CURVATURE PROBLEM

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Abstract. We develop a finite element method for solving the Dirichlet problem of the one-dimensional prescribed curvature equation due to its irreplaceable role in applications. Specifically, we first analyze the existence and uniqueness of the solution of the problem and then develop a finite element method to solve it. The well-posedness of the finite element method is shown by employing the Banach fixed-point theorem. The optimal error estimates of the proposed method in both the H^1 norm and the L^2 norm are established. We also design a Newton type iteration scheme to solve the resulting discrete nonlinear system. Numerical experiments are presented to confirm the order of convergence of the proposed method.

Key words. Prescribed curvature equation, finite element method, Newton iteration, Banach fixed-point theorem.

1. Introduction

The purpose of this paper is to develop a finite element method for solving the Dirichlet problem of the one-dimensional prescribed curvature equation. The study of the prescribed curvature equation originates from Thomas Young's [35] and Pierre-Simon Laplace's [24] independent research about the properties of capillary surfaces which date back to 1805. Its mathematical theory was built by Gauss [16] in 1830, and was enriched by numerous researchers [17, 28]. At the present time, there is still great interest in the study of this equation including the one dimensional case [19, 23, 29] and high dimensional cases [17, 18, 32]. The prescribed curvature equation appears in many important fields including classical problems in differential geometry (e.g. minimal surfaces [26, 28]; constant curvature surfaces [28]) and the static fluid problem in fluid mechanics such as the Young-Laplace equation [15, 35]. In particular, the one-dimensional equation plays an irreplaceable role in applications such as modeling corneal shape [10, 27, 33] and modeling electrostatic micro-electro mechanical systems [6, 7, 13].

The interest in the one-dimensional prescribed curvature equation has led to much progress in analyzing the existence, non-existence and multiplicity of its solutions. Studying the equation was inspired by an open problem proposed by Haim Brezis et al. in [1] for investigating the multiplicity and structure of the solution of a specific semilinear elliptic problem related to a simplified version of the equation. The equation under study has more severe nonlinearity in its operator and more complexity of the multiplicity of its solution. There exist a large number of papers which focused on the existence of solutions of the equation by using the barrier method [23], the time map method [29] and the sub-super solution method [25]. Especially, the equation with a general forcing term that depends on the unknown solutions and their gradient was considered in [3, 29]. Moreover, some fascinating aspects of the Dirichlet problem of the one-dimensional prescribed curvature equation were obtained in [30, 31], including the disappearing solution behavior and the bifurcation property of the solution. For the computational issue of this problem,

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the shooting method was studied in [2], the finite difference method was investigated in [8] and the conjugate gradient method was considered in [20]. These methods are difficult to be extended to a higher dimensional case. Specifically, the shooting method converts the original problem to an equivalent initial value problem involved in many parameters. The finite difference method for solving this problem in higher dimensions suffers from the difficulty in handling the curve boundaries imposed the Dirichlet boundary condition and as a result it is difficult to extend it to a higher dimensional case. The conjugate gradient method treats the original problem as an equivalent minimization problem, and it is difficult to extend it reliably and easily to high dimensions for more complicated and important case.

At present, the finite element method are used for special two dimensional cases of the prescribed curvature problem. In [21], Johnson and Thomée developed the finite element error analysis to obtain the optimal H^1 and $L^p, 1 \leq p < 2$ estimates for the minimal surface problem by using the piecewise linear approximate functions. A posteriori error analysis for the Dirichlet problem of the prescribed mean curvature equation with homogeneous boundary conditions was developed in [14] and the finite element method for the discrete Plateau's problem and the corresponding H^1, L^2 error estimates were obtained in [11, 12] (see also the references cited therein) by dealing with the equivalent energy functional. In the aforementioned finite element methods, the strict convexity of the corresponding energy functional provides conveniences for the study of the existence and uniqueness of the approximate solution of the specific two dimensional cases. However, for the one-dimensional prescribed curvature equation, when the forcing term depends on the unknown solution, the finite element method may result in a nonconvexity of the corresponding energy functional. This requires a new approach to deal with the existence and uniqueness of the approximate solution of the one-dimensional case. We shall accomplish this by applying the Banach fixed point theorem. Therefore, the finite element method for one-dimensional problem deserves further investigation.

Our goal is to develop a finite element method for solving the Dirichlet boundary value problem of the one-dimensional prescribed curvature equation, with potential of easy extension to handle cases where the forcing term depends on both the unknown solution and its gradient and handle the high dimensional case. We shall adopt the standard Lagrange finite element for this purpose. As explained in [4], advantages of using the standard Lagrange finite element include the simplicity of its implementation and the ability of handling the general case in which the forcing term may depend on the unknown solution and its gradient. For simplicity of presentation, we consider in this paper the equation with the forcing term independent of the unknown solution. The method developed in this paper can be easily extended to the general case. Specifically, we establish the existence and uniqueness of the solution for this problem by using the shooting method. We study the regularity of the solution, which lays a foundation for the convergence analysis of the proposed method. We construct the finite element scheme by using the simple and practical Lagrange finite element so that its discrete linearization is consistent with the linearization of the original nonlinear equation. We identify a fixed point of the constructive nonlinear operator in a small ball by using the Banach fixed-point theorem to simultaneously show the well-posedness of the finite element method and derive an optimal H^1 error estimate. Furthermore, we obtain the optimal L^2 error estimate by extending the Nitsche strategy naturally within our framework.

A critical issue in analyzing the proposed method for this problem is the nonlinearity of the differential operator involved in the equation. To overcome this