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A 3D CONFORMING-NONCONFORMING MIXED FINITE ELEMENT FOR SOLVING SYMMETRIC STRESS STOKES EQUATIONS

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Abstract. We propose a 3D conforming-nonconforming mixed finite element for solving symmetric stress Stokes equations. The low-order conforming finite elements are not inf-sup stable. The low-order nonconforming finite elements do not satisfy the Korn inequality. The proposed finite element space consists of two conforming components and one nonconforming component. We prove that the discrete inf-sup condition is valid and the discrete Korn inequality holds uniformly in the mesh-size. Based on these results we give some numerical verification. In addition, this element is compared numerically with six other mixed finite elements.

Key words. Mixed finite element, symmetric stress, Korn's inequality, Stokes equations.

1. Introduction

Finite element methods for the 2D symmetric stress Stokes problem have been extensively studied in the literature, and most of stable schemes are summarized in book [6]. However, only little attention has been paid to the 3D problems. Actually, the nonconforming elements of Crouseix-Raviart [10] is only suitable if $\Gamma_N = \emptyset$ due to a missing Korn's inequality in two as well as in three dimensions. In 2D, the nonconforming elements of Kouhia and Stenberg [17] circumvent this problem by choosing one component nonconforming element and the other one conforming element. [19] has also given a counter example to show that if both of the two components are nonconforming rotated Q_1 elements, discrete Korn's inequality is invalid. From these works, we are hinted to use different spaces for different components of the velocity to assure the well-posedness of the discrete problem. We prove that the mixed finite elements with one nonconforming component, the nonconforming rotated Q_1 element, the conforming Q_2 element and the conforming Q_1 element for the other two components of the velocity, respectively, satisfy the discrete Korn inequality. In addition, such a velocity element combined with a piecewise constant pressure element, i.e., $RQ_1 \times Q_2 \times Q_1$-$P_0$, is inf-sup stable. For the nonconforming rotated Q_1 element, Rannacher and Turek analyzed this element in [23] for solving the (gradient) Stokes equations. The element has shown superconvergence in special meshes according to [18]. However, as mentioned above, this rotated Q_1 element does not satisfy the discrete Korn inequality and does not solve the symmetric stress Stokes equations (see numerical tests below.) Similarly, the Cai-Douglas-Santos-Sheen-Ye's element [7, 8, 11] does not work for the symmetric stress Stokes equations either. But the element can be used for one component of the velocity, replacing the rotated Q_1 element.

We note that, unlike the 2D case, two components of C^0-Q_1 of the velocity are not enough. That is, the $RQ_1 \times Q_1 \times Q_1$-$P_0$ mixed finite element does not solve the symmetric stress Stokes equations. A numerical test on the element is provided. The proposed $RQ_1 \times Q_2 \times Q_1$-$P_0$ is almost the simplest mixed finite

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element. Here we can have another slightly simpler version of the proposed mixed finite elements that the conforming Q_2 can be replaced by the conforming $Q_{2,1,1}$ space, for example, where $Q_{1,2,1}$ denote the polynomials of separated degrees 2, 1 and 1. The analysis for this mixed finite element is same. Numerically, we test the newly proposed finite element, along with six other typical mixed finite elements, including this simplified element $RQ_1 \times Q_{2,1,1} \times Q_1$-$P_0$.

The rest of the paper is organized as follows. In section 2, we present the symmetric stress Stokes problem. In section 3, we define the conforming nonconforming combined mixed finite element. The well-posedness of the discrete problem and an error estimate will be proved for the proposed mixed finite element. Section 4 concludes this paper with seven numerical experiments.

Throughout this paper, standard notation on Lebesgue and Sobolev spaces is employed. $(\cdot,\cdot)$ denotes the L^2 scalar product over Ω. Let $\|\cdot\|_{0,\Omega}$ denote the L^2 norm over a set $\Omega \subset \Omega$ and $\|\cdot\|_0$ abbreviate $\|\cdot\|_{0,\Omega}$. $|\cdot|_{1,h}$ denotes the semi-H^1 norm for nonconforming functions and $|\cdot|_1$ the standard semi-H^1 norm. $\partial\Omega$ denotes the boundary of Ω. If there is no special instruction, the bold face letter will indicate a vector or vector space in order to distinguish it from scalars. Let $A \lesssim B$ abbreviates that there is some mesh-size independent generic constant $0 \le \mathcal{C} \le \infty$ such that $A \le \mathcal{C}B$.

2. The symmetric stress Stokes problem

Assuming the domain $\Omega \in R^3$ is a convex, polyhedral, bounded Lipschitz domain, which can be triangulated by parallelepipeds (or simply by cuboids), with closed Dirichlet boundary Γ_D and Neumann boundary $\Gamma_N = \partial\Omega \setminus \Gamma_D$, both with nonzero two dimensional measure, and some right-hand side functions $\mathbf{f} \in [L^2(\Omega)]^3$, $\mathbf{u}_1 \in [H^{3/2}(\Gamma_D)]^3$ and $\mathbf{g} \in [H^{1/2}(\Gamma_N)]^3$, the three dimensional symmetric stress Stokes problem seeks the velocity $\mathbf{u} \in [H^1(\Omega)]^3$ and pressure $p \in L^2(\Omega)$ such that

$$
(1) \qquad \begin{cases} -2\mu \operatorname{div} \varepsilon(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_D & \text{on } \Gamma_D \\ \sigma\mathbf{n} = \mathbf{g} & \text{on } \Gamma_N, \end{cases}
$$

where, and throughout this paper, $\mathbf{n}$ is the unit normal vector on the boundary, $\mu > 0$ is the viscosity, $\sigma = (2\mu\varepsilon(\mathbf{u}) - pI)$, and $\varepsilon(\mathbf{u})$ is the symmetric gradient of a vector, which is

$$
\begin{aligned}
\varepsilon(\mathbf{u}) &= \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \\
&= \frac{1}{2}\begin{pmatrix} 2\partial_x u_1 & \partial_y u_1 + \partial_x u_2 & \partial_z u_1 + \partial_x u_3 \\ \partial_y u_1 + \partial_x u_2 & 2\partial_y u_2 & \partial_z u_2 + \partial_y u_3 \\ \partial_z u_1 + \partial_x u_3 & \partial_z u_2 + \partial_y u_3 & 2\partial_z u_3 \end{pmatrix}
\end{aligned}
$$

for any $\mathbf{u} = [u_1\ u_2\ u_3]^T \in [H^1(\Omega)]^3$.

We note that due to the boundary conditions, the symmetric stress Stokes problem (1) is not equivalent to a standard (gradient) Stokes problem.

The weak formulation of equation (1) reads Find $(\mathbf{u}, p) \in \mathbf{V} \times L^2(\Omega)$, such that

$$
(2) \qquad \begin{cases} a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) = (\mathbf{f},\mathbf{v}) + \displaystyle\int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,ds & \forall \mathbf{v} \in \mathbf{V}_0, \\ b(q,\mathbf{u}) = 0 & \forall q \in L^2(\Omega), \end{cases}
$$