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A NODAL SPARSE GRID SPECTRAL ELEMENT METHOD FOR MULTI-DIMENSIONAL ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We develop a sparse grid spectral element method using nodal bases on Chebyshev-Gauss-Lobatto points for multi-dimensional elliptic equations. Since the quadratures based on sparse grid points do not have the accuracy of a usual Gauss quadrature, we construct the mass and stiffness matrices using a pseudo-spectral approach, which is exact for problems with constant coefficients and uniformly structured grids. Compared with the regular spectral element method, the proposed method has the flexibility of using a much less degree of freedom. In particular, we can use less points on edges to form a much smaller Schur-complement system with better conditioning. Preliminary error estimates and some numerical results are also presented.

Key words. Sparse grid, spectral element method, high-dimensional problem, adaptive method.

1. Introduction

Many scientific and engineering applications require solving high-dimensional partial differential equations (PDEs), e.g. the electronic Schrödinger equation [15], the Black-Scholes equation for option pricing [4], etc. Traditional methods for solving high-dimensional PDEs usually use tensor-product discretizations which need N^d total points if N points are used in each dimension. Such approximation quickly become unfeasible for problems with moderate to large dimensions, due to the so-called curse of dimensionality. However, for functions with special regularity, one can use sparse grid or hyperbolic cross approximation. This approach is first introduced by Smolyak for high-dimensional quadrature and interpolation problems in 1963 [20], and then extended for solving PDEs by Bungartz, Griebel et al. [6, 10, 7, 11, 2], where lower order finite element bases is used for problem with non-periodic boundary conditions and Fourier or wavelets bases are used for problem with periodic boundary conditions. For non-periodic spectral sparse grids, Barthelmann et al. [3] gave an error estimate for interpolation based on Chebyshev sparse grid, Shen and Wang [17] analyzed the approximation error of using hyperbolic cross Legendre approximation for solving standard elliptic PDEs. Shen and Yu developed efficient algorithms using sparse grid based on Chebyshev-Gauss-Lobatto (CGL) points for solving elliptic PDEs with non-periodic boundary conditions in bounded domains [18] and unbounded domains [19].

Even though sparse grids are developed for dealing with high dimensional problems, they are most effective for problems with some special properties. For example, Yserentant proved that the wave function of electronic Schrödinger equation has bounded mixed derivatives, which can be well approximated by sparse grids

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[21, 22]. However, many factors can contribute to the singular behavior of solutions. To deal with these and more general singularities, we construct in this paper a sparse grid Spectral Element Method (sgSEM) for high-dimensional PDEs. By decomposing the computational domain into sub-domains, we can effectively deal with many types of singularities, such as those in initial conditions, forcing terms and coefficients of the PDEs, and achieve a better convergence rate than usual Sparse Grid Methods (SGM) and Spectral Element Methods (SEM).

The rest of this paper is organized as follows. In Section 2, we briefly review some basic setup and results of the usual sparse grid methods. Then in Section 3, we construct the sgSEM for a model elliptic equation with variable coefficients and describe its basic properties. In Section 4, we develop a modified sgSEM (which we name it sgSEMm) with less nodes on edges, so as to reduce the size and condition number of the Schur-complement for the resulting linear algebraic system. Preliminary error estimates of sgSEM is given in Section 5 and numerical results for several examples are presented in Section 6 to verify the convergence and efficiency of the proposed methods. We end the paper with a few concluding remarks.

2. A brief review of sparse grid spectral methods

In order to present our sparse grid spectral element method, we need to first introduce the sparse grid spectral method. We start with some notations. We use bold letters to denote a d-dimensional vector. For d-dimensional coordinates, we use superscripts, e.g., $\boldsymbol{x} = (x^1, \dots, x^d)$, to denote its components, and use subscripts, e.g., x_j, to denote interpolation points. For d-dimensional vectors other than coordinates, we use subscripts to denote its components. When $d = 1$, we have $x = x^1$.

2.1. Sparse grids and fast spectral transforms on sparse grids.

Sparse grid method provides a feasible approximation for some high-dimensional functions. To introduce the sparse grid, we consider approximation of a high dimensional function $f(\boldsymbol{x}) : I^d \to \mathbb{R}$, where $I := [-1, 1]$. Suppose we have a one-dimensional interpolation scheme on a set of nested grids $\mathcal{X}_l : l = 0, 1, \dots$, where $\mathcal{X}_l \subset \mathcal{X}_{l'}$, if $l < l'$. Denote $m_l = \text{Card}\{\mathcal{X}_l\}$ and

$$\mathcal{X}_l = \{\, x_j : j \in \mathcal{I}_l \,\}, \quad \mathcal{I}_l := \{\, 0, \dots, m_l - 1 \,\}. \tag{1}$$

The interpolation of level l is defined as

$$\mathcal{U}_l(f) := \sum_{k \in \mathcal{I}_l} \hat{f}_k^l \phi_k(x), \quad s.t. \;\; \mathcal{U}_l(f)(x_j) = f(x_j), \quad \forall j \in \mathcal{I}_l, \tag{2}$$

where $\phi_k(x), k \in \mathcal{I}_l$ are the basis functions of interpolation space $V_l = \text{span}\{\phi_k : k \in \mathcal{I}_l\}$ and $\hat{f}_k^l, k \in \mathcal{I}_l$ are the spectral coefficients. The above scheme is well defined if the matrix $\Psi = \big(\phi_k(x_j)\big)_{k,j \in \mathcal{I}_l}$ is non-singular.

If one extends the one-dimensional scheme to d-dimension by tensor product rule, then the total number of points will be $[m_l]^d$ which grows exponentially with d, leading to the so called curse of dimensionality. To reduce the number of points, Smolyak [20] introduced the so-called sparse grid approximation:

$$\mathcal{U}_l^d(f) := \sum_{|\boldsymbol{l}|_1 \leqslant l} \hat{\mathcal{U}}_{l_1} \otimes \hat{\mathcal{U}}_{l_2} \otimes \cdots \otimes \hat{\mathcal{U}}_{l_d}(f), \tag{3}$$