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A PRIORI ERROR ANALYSIS OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE VISCOUS BURGERS-POISSON SYSTEM

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Abstract. In this paper, we propose and analyze the local discontinuous Galerkin method for the viscous Burgers-Poisson system. The proposed method preserves two invariants and hence, yields solutions even for long time. *A priori* error estimates, which are of order $\mathcal{O}(h^{k+1})$, when polynomials of degree $k \geq 1$ are used for approximating solutions are established. Finally, numerical experiments are conducted to confirm our theoretical results.

Key words. Burgers-Poisson system, local discontinuous Galerkin method, *A priori* error estimates.

1. Introduction

We consider the following coupled system of viscous Burgers and Poisson equations: find a pair of solutions (u, ϕ) such that

$$u_t + (\frac{u^2}{2} - \phi)_x - \epsilon u_{xx} = 0, \qquad x \in [0, L] = I, \quad t > 0, \tag{1}$$

$$\phi_{xx} - \phi = u, \tag{2}$$

with $\epsilon > 0$ and periodic boundary conditions:

$$\begin{aligned} &u(t, L) = u(t, 0),\ u_x(t, L) = u_x(t, 0) \text{ and} \\ &\phi(t, L) = \phi(t, 0),\ \phi_x(t, L) = \phi_x(t, 0),\ \text{for } t > 0, \end{aligned} \tag{3}$$

and initial condition:

$$u(0, x) = u_0(x), \qquad x \in I. \tag{4}$$

This problem is one dimensional version of the Navier-Stokes-Poisson system, which often models the transport of charged particles under the influence of the self-consistent electro-static potential as a force arising in the study of collision of dusty plasma, see [7], [9]. This system admits conservation of momentum and L^2 *a priori* bound. Global existence of weak solutions to the Navier-Stokes-Poisson system with large initial data has been proved by Donatelli [6] using Galerkin method and P.L.Lions theory, [13]. Without much difficulty, this theory can be extended to include the global existence of a unique solution for the Burgers-Poisson system (1)-(4).

In recent years, Discontinuous Galerkin (DG) methods are becoming popular due to their flexibility in local mesh adaptivity, element wise conservative property and in taking care of nonuniform degrees of approximation of the solution whose smoothness may exhibit a wide variation over the computational domain. These methods are using completely discontinuous piecewise-polynomials for the numerical solution and the test functions. These schemes are first proposed for solving first order PDEs such as nonlinear conservation laws, [14], [1], [2], [3], [4]. The

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local discontinuous Galerkin (LDG) method is an extension of DG methods for solving higher order PDEs. It was first designed for convection-diffusion equations in [5], and has been extended to other higher order wave equations, including the KdV equation, [19], [16], [11], [17], see, also the recent review paper [18] on the LDG methods for higher order PDEs. The idea of the LDG method is to rewrite higher order equations into a first order system, and then apply DG schemes on the system with appropriate choices of numerical fluxes. Related to our problem, a LDG method was proposed in [12] for the inviscous Burgers-Poisson equation. This scheme preserves the mass and energy of the smooth solution and was proven to be optimal convergence for k even.

In this article, LDG method is applied to the viscous Burgers-Poisson system (1)-(4). Then, it is observed that the semidiscrete system preserves two invariants and as a result, we prove *a priori* bounds in $L^\infty(L^2)$ for the discrete solutions. It is, further, shown that rate of convergence is of order $k+1$ for approximate solution u_h, when polynomial of order k is used to approximate u. The generalized numerical fluxes, which depend on a parameter $\theta \in [0, 1/2]$ are used in the proposed scheme. For $\theta = 1/2$, it is noted that the order of convergence is optimal as in [12] for even degree polynomial degrees. When $\theta \in [0, 1/2)$, optimal error estimates are derived, but with constants in the error analysis explicitely depend on $1/\sqrt{\epsilon}$, where ϵ is a viscosity parameter.

We use standard notation for norms and seminorms in Sobolev spaces. Say for example, for any integer $m \geq 0$, we denote by $H^m(I)$, the Hilbert Sobolev space with norm $\|\cdot\|_m$ and seminorm $|\cdot|_m$. We also use the spaces $L^p(0,T;H^m(I)),\ 1 \leq p \leq \infty$ as the spaces of functions v such that $\int_0^T \|v(s)\|^p_{H^m(I)}\ ds < \infty$. Denote by C a positive generic constant, which does not depend on the mesh parameters, but may vary from context to context in the text.

2. Conservation Properties and *A Priori* Bounds

This section deals with some conservation properties and *a priori* bounds for the viscous Burgers-Poisson system (1)-(4).

Theorem 2.1. *Let* (u, ϕ) *be a pair of solutions of the coupled system* (1)-(4). *Then the following conservation property holds:*

$$\int_0^L u(x,t)dx = \int_0^L u_0(x)\ dx. \tag{5}$$

Further, u *satisfies*

$$\int_0^L |u(x,t)|^2 dx \leq \int_0^L |u_0(x)|^2\ dx. \tag{6}$$

Proof. Integrating equation (1) with respect to space variable x yields

$$\int_0^L u_t\ dx + \int_0^L (\frac{u^2}{2})_x\ dx - \int_0^L \phi_x\ dx - \epsilon \int_0^L u_{xx}\ dx = 0,$$

which can be rewritten using periodic boundary conditions as

$$\frac{d}{dt}\int_0^L u(t,x)\ dx = 0.$$

Integrating above equation with respect to time t yields the equation of conservation of momentum, that is,

$$\int_0^L u(t,x)\ dx = \int_0^L u(0,x)\ dx = \int_0^L u_0(x)\ dx. \tag{7}$$