INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 6, Pages 842–868

VARIABLE TIME-STEP θ-SCHEME FOR NONLINEAR SECOND ORDER EVOLUTION INCLUSION

KRZYSZTOF BARTOSZ

Abstract. We deal with a multivalued second order dynamical system involving a Clarke subdifferential of a locally Lipschitz functional. We apply a time discretization procedure to construct a sequence of solutions to a family of the approximate problems and show its convergence to a solution of the exact problem as the time step size vanishes. We consider a nonautonomous problem in which both the viscosity and the multivalued operators depend on time explicitly. The time discretization method we use, is the θ-scheme with $\theta \in [\frac{1}{2}, 1]$, thus, in particular, the Crank-Nicolson scheme and the implicit Euler scheme are included. We apply our result to a class of dynamic hemivariational inequalities.

Key words. Clarke subdifferential, hemivariational inequality, second order inclusion, time discretization, numerical methods.

1. Introduction

In this paper we deal with a time discretization method for a second order, dynamic subdifferential inclusion, involving nonlinear, time dependent viscosity operator and a multivalued term that is a Clarke subdifferential of a locally Lipschitz continuous function that is possibly nonmonotone and nonsmooth.

Various types of Clarke subdifferential inclusions, formulated often in an alternative way, by means of hemivariational inequalities (HVIs), are motivated by numerous physical phenomena, in which contact problems in mechanics play a leading role. Indeed, the number of applications became a main impulse for research in this field. After first results of Clarke and Panagiotopoulos (see [10, 29]), the theory of HVIs has been developed by Miettinen, Migórski, Motreanu, Naniewicz, Panagiotopulos and Ochal (see [21, 22, 23, 24, 25, 27, 28, 30]). Currently many authors devote their attention either to the theory of HVIs (see for instance [8, 9, 14, 19]) or to its applications in modelling of contact problems in mechanics (see [4, 5]). For the present state of the art we refer to [6, 26]. In spite of an impressive progress of the theory, there are still relatively few results concerning numerical methods for HVIs and many problems still remain open in this field. In particular, the dynamic development of computational devices, allows to implement increasingly complex mathematical models. Due to this fact, the numerical results become more and more needed in case of HVIs as well. Haslinger and Miettinen were the first to apply the Finite Element Methods for problems modelled by HVIs (see [15, 20]). As for the time discretization methods in HVIs, the strong results concerning parabolic problems, were obtained by Kalita et al. in [7, 16, 17, 18]. Similar methods have been used by Liu, Peng and Xiao in [31, 32, 33] in the case of evolution HVIs with doubly nonlinear operators. In particular the second order HVIs has been studied in [33] in the framework of the Gelfand triple $V \subset H \subset V^*$, where the multivalued term is defined on the space H. In this paper, the multivalued term is defined on another Banach space U such that there exists a linear, continuous and compact operator $\iota : V \to U$. In applications $U = L^p(\Gamma; \mathbb{R}^d)$, where Γ is contained in the

Received by the editors December 13, 2015.
2000 *Mathematics Subject Classification.* 35G55, 74H15, 74H20, 65J15, 35R70.

boundary of the set $\Omega \subset \mathbb{R}^d$. It allows to apply our result to HVIs arising from nonmonotone and nonsmooth contact problems in mechanics. For other recent results concerning numerical methods for static or dynamic HVIs, we refer for instance to [2, 3, 11, 12, 13, 35].

In this work, we deal with a numerical analysis of dynamic, second order inclusion, which is based on time semidiscrete θ-scheme. To this end, we apply a technique, that was used in [18] in a study of parabolic problems. We apply our result to a class of dynamic boundary HVIs. In several ways, our paper improves known existence results in this area. One of basic applications of HVI's is mathematical modelling of a behaviour of physical body which occupies a region $\Omega \in \mathbb{R}^d$ and stays in a contact with a foundation. It is usually assumed, that the body is clamped on a part of boundary $\Gamma_D \subset \partial\Omega$. Our result allows to skip this restriction (see Section 7). It generalizes also [22] by removing a smallness assumption for $p = 2$ (see Remark 32 for details). Moreover, not only we provide the existence result, but we also construct a sequence of functions, which approximate the solution of the exact problem. From this point of view, our method is not only constructive but can be used in computer implementation. Finally, the present paper generalizes also the result obtained by the author in [6] for the autonomous case.

The rest of the paper is organized as follows. In Section 2, we introduce the notations and definitions used in the paper and present several auxiliary propositions. In Section 3, we formulate an abstract second order subdifferential inclusion and describe assumptions on the data of the problem. We also provide two crucial lemmas concerning properties of the Nemytskii operator corresponding to the viscosity operator A. In Section 4, we formulate an auxiliary discrete problem and provide its solvability. Based on this, we construct a sequence of piecewise constant and piecewise linear functions, for which we derive a-priori estimates and, using the reflexivity of the spaces, we obtain a weak convergence result. Finally, we provide the main existence result, namely, we show, that the limit function is a solution of the exact problem. In particular, this provides a constructive proof of the existence for the problem. In Section 5, we use the (S_+) property of the viscosity operator in order to obtain a strong convergence result. In Section 6, we apply the abstract result to a class of boundary HVIs arising from contact problems in mechanics. In Section 7, we deal with the non-clamped dynamic contact problem modelled by HVI's.

2. Preliminaries

In this section we recall some definitions and propositions which we will refer to in the sequel. We start with the definitions of Clarke directional derivative and Clarke subdifferential. Let X be a real Banach space, X^* its dual and let $J\colon X \to \mathbb{R}$ be a locally Lipschitz functional.

Definition 1. *Generalized directional derivative in the sense of Clarke at the point $x \in X$ in the direction $v \in X$, is defined by*

$$J^0(x, v) = \limsup_{y \to x, \lambda \searrow 0} \frac{J(y + \lambda v) - J(y)}{\lambda}. \tag{1}$$

Definition 2. *Clarke subdifferential of J at the point $x \in X$ is defined by*

$$\partial J(x) = \{\xi \in X^* \mid J^0(x, v) \geq \langle \xi, v \rangle_{X^* \times X} \text{ for all } v \in X\}.$$