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A NOVEL ADAPTIVE FINITE VOLUME METHOD FOR ELLIPTIC EQUATIONS

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Abstract. In this paper, we propose a novel adaptive finite volume method (AFVM) for elliptic equations. As a standard adaptive method, a loop of our method involves four steps: $Solve \rightarrow Estimate \rightarrow Mark \rightarrow Refine$. The novelty of our method is that we do not have the traditional "completion" procedure in the *Refine* step. To guarantee the conformity, a triangular element with a hanging node is treated as a quadrilateral element, and the corresponding function space consists of the bilinear functions. The optimal computational complexity of our AFVM is validated by numerical examples.

Key words. Adaptive finite volume method, hanging nodes, hybrid meshes, error analysis.

1. Introduction

The finite volume method (FVM) is a popular numerical tool for partial differential equations (PDEs), cf. [1, 2, 5, 13–18, 20–22, 24–26, 28, 29, 31–33, 37]. Recently, the adaptive finite volume method (AFVM) attracts a lot of attention, see [6, 8, 11, 12, 19, 23, 27, 30, 36]. Especially the a *posteriori* error estimator has been studied in many papers, see [36] for the hierarchical type and [6, 11, 12, 23, 27, 30] the residual type error estimators.

In this paper, we design a novel AFVM for elliptic equations. Unlike the previous works which pay a lot of attention on the construction of a *posteriori* error estimators for the FVM, here we construct our novel adaptive method modifying the adaptive strategy. It is known that an iteration of a standard adaptive method involves four steps: $Solve \rightarrow Estimate \rightarrow Mark \rightarrow Refine$. In particular, in the *Refine* step, after having refined the mesh according to the previous marking step, we should refine more elements to eliminate the so called "hanging nodes". The novelty of our method is that we do not have the traditional "completion" procedure in the *Refine* step which allow hanging nodes. In order to guarantee the conformity, a triangular element with one hanging node will not be divided if the edge with the hanging node is not a base of the triangle. We consider this triangle with the hanging node as a quadrilateral. We only divide a triangular element has a hanging node on the base, or has more than one hanging nodes on edges. As a consequence, our meshes in AFVM consist of hybrid triangular and quadrilateral elements, to ensure the continuity of our trial function of the finite volume scheme in the *Solve* step, we let the trial function on triangular element be constructed as the linear function and on quadrilateral element (a triangular element with a hanging node) as the isoparametric bilinear. In other words, we define the trial function as

$$\begin{cases} \text{linear function,} & \text{if a triangular element has no hanging node} \\ \text{isoparametric bilinear function,} & \text{if a triangular element has a hanging node} \end{cases}.$$

We follow the Zienkiewicz-Zhu [34, 35] type gradient recovery operator in the *Estimate* step, and the marking strategy proposed by Dörfler [10] in the *Mark* step.

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One may naturally ask a question: how about the actual significance of the novel AVFM? It is known that to keep the conformity, in the *Refine* step, one often needs an additive "completion" procedure [3] to eliminate the hanging nodes. Our numerical experiments show that the novel AFVM decrease the steps of bisection for the conformity compare to the standard AFVM which needs traditional "completion" procedure. Moreover, our AFVM possesses the local conservation property. Furthermore, suppose u is the exact solution of the elliptic equation, our numerical results show that

$$|u-u_k|_1 \leq C_1 N_k^{-1/2} \text{ and } \|u-u_k\|_0 \leq C_2 N_k^{-1},$$

where u_k and N_k are the FVM solution and the number of elements of the mesh to k-th iteration, C_1 and C_2 are two constants. We note that in our numerical example, the optimal convergence order of H^1 and L^2 errors can be obtained even if $u \in H^{1+\frac{2}{3}-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ and $u \notin H^2(\Omega)$, and the domain Ω is not convex.

The main idea of our AFVM can be applied to general elliptic equations. To illustrate the basic idea, we focus on the model problem

$$\begin{aligned} -\nabla \cdot (\alpha \nabla u) &= f \quad \text{in } \Omega, && (1)\\ u &= 0 \quad \text{on } \partial\Omega, && (2) \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a convex polygon domain, α is a piecewise continuous function that bounded below: There exists a positive constant $\alpha_0 > 0$ such that $\alpha(x) \geq \alpha_0$ for almost all $x \in \Omega$, and f is a real valued function defined on Ω. The stability analysis of our finite volume scheme is a routine under the framework of linear on triangular element [29] and isoparametric bilinear on quadrilateral element [25], and we obtain the optimal convergence rate of H^1 and L^2 norms.

The rest of this paper is organized as follows. In the next section we introduce a FVM on hybrid triangular and quadrilateral meshes. The optimal convergence properties are studied both in H^1 and L^2 spaces. Our novel AFVM is presented in Section 3. Numerical examples are provided in Section 4 to validate that our novel AFVM has optimal computational cost and the theoretical results of our FVM.

We end this section with some notations. For an integer $m \geq 0$ and $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ denote the standard Sobolev spaces of functions that have generalized derivatives up to order m in $L^p(\Omega)$. The norm (or semi-norm) is defined by $\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_p^p\right)^{1/p}$ (or $|u|_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|_p^p\right)^{1/p}$) for $1 \leq p < \infty$, and with the standard modification for $p = \infty$. $H^m(\Omega) := W^{m,2}(\Omega)$ and $H_0^1(\Omega)$ denote the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$. For simplicity, in the rest of the paper we will omit the subscript index when $p = 2$ and the domain index Ω if needed. Furthermore, $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$-inner product. To avoid writing constants repeatedly, "$A \lesssim B$" means that A can be bounded by B multiplied by a constant which is independent of the parameters which A and B may depend on, "$A \gtrsim B$" means that B can be bounded by A. "$A \sim B$" means "$A \lesssim B$" and "$B \lesssim A$".

2. A finite volume method on hybrid meshes

2.1. A hybrid triangular and quadrilateral mesh. We partition Ω into a mesh $\mathcal{T}_h$ consisting of a finite number of triangles and convex quadrilaterals, where h is the largest diameter of all triangles and quadrilaterals, and we call $\mathcal{T}_h$ the primal mesh of Ω, see Fig.1. We denote by $\mathcal{N}_h$ the set of all vertices of $\mathcal{T}_h$, and let $\mathcal{N}_h^\circ = \mathcal{N}_h \backslash \partial\Omega$ be the set of all interior vertices.