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A NEW METHOD FOR SIMULTANEOUSLY RECONSTRUCTING THE SPACE-TIME DEPENDENT ROBIN COEFFICIENT AND HEAT FLUX IN A PARABOLIC SYSTEM

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This paper is dedicated to the memory of Professor Mahmoud Hashem Farag

Abstract. This paper studies a regularization approach for simultaneously reconstructing space-time dependent Robin coefficient $\gamma(\mathbf{x}, t)$ and heat flux $q(\mathbf{x}, t)$. The differentiability results and adjoint systems are established. A standard finite element method (FEM) is employed to discretize the constrained optimization problem which is reduced to a sequence of unconstrained optimization problem by adding regularization terms. We propose an improved algorithm for the introduced problem based on modified conjugate gradient method (MCGM) for quadratic minimization. Numerical experiments present the efficiency, accuracy, and robustness of the proposed algorithm.

Key words. Simultaneous identification, numerical reconstruction, Robin coefficient and heat flux, Tikhonov regularization, FEM, MCGM.

1. Introduction

The inverse problem arising in reconstructing the heat transfer coefficient, called Robin coefficient $\gamma(\mathbf{x}, t)$, which represents the convection between the conducting body and the ambient environment from the boundary measurements of the solution and space-time dependent heat flux $q(\mathbf{x}, t)$. Háo [8] determined the space or time dependent heat transfer coefficient using nonlinear conjugate gradient method combined with a boundary element direct solver. In many distributed parameter identification problems, the Robin inverse problem suffers from ill-posedness, such as, small error in the data which leads to large deviations in the solution. Therefore, specialized techniques are necessary to keep the stability in the solution. Numerically, an engineering approach has been applied to estimate the time-dependent Robin coefficient from the measured temperature data for quenching process [1], using the sequential function specification method [2]. However, the approach is generally influenced by the noise in the data and then the accuracy of the solutions. Another popular engineering approach is the variational method [14, 23]. Marián [19] studied the recovery of a time-dependent Robin coefficient in a semilinear parabolic equation from an over-specified nonlocal boundary conditions and proposed a temporal discretization based on Rothes method with some convergence analysis. However, the spatial discretization that is necessary for practical computations was not considered.

Slodička, et al. [18] introduced a mathematical analysis for the estimation of the time-dependent Robin coefficient in a nonlinear boundary condition for one-dimensional heat equation, and showed the existence and uniqueness of the solution. Deng, et al. [3, 4] introduced a two-level space-time domain decomposition method for solving an inverse source problem associated with the time-dependent convection-diffusion equation in three dimensions. Jiang, et al. [11] proposed an

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efficient overlapping domain decomposition method for solving some typical linear inverse problems, including identification of the flux, source strength, and initial temperature in second order elliptic and parabolic systems. Jin and Lu [13] studied the space-time dependent Robin coefficient in one and two dimensional cases using nonlinear conjugate gradient method. Xie and Zou [21] introduced the mathematical and numerical justification of regularization approaches for reconstructing the heat flux in both space and time with investigation of a FEM. Jiang and Talaat [12] focused on the numerical simultaneous reconstruction of the spatially-dependent Robin coefficient and heat flux using Levenberg-Marquardt and surrogate functional method. The present study focuses on reconstructing the space-time dependent Robin coefficient and heat flux, simultaneously using MCGM for solving the nonlinear inverse problem.

The rest of this paper is organized as follows: Section 2 describes the mathematical and variational formulation of the problem. Section 3 investigates the Tikhonov regularization approach in the case that $\gamma(\mathbf{x},t)$ and $q(\mathbf{x},t)$ are unknowns parameters. Section 4 derives the differentiability results to find the gradient formula with respect to $\gamma(\mathbf{x},t)$ and $q(\mathbf{x},t)$. Also, introduces the adjoint problems to find explicit relations that simplify computing the step lengths. Section 5 introduces the FEM and its convergence analysis. Section 6 introduces the numerical algorithm based on the MCGM. Section 7 presents the numerical experiments to investigate the efficiency, accuracy, and robustness of the proposed algorithm. Finally, we draw the conclusion and the future work in section 8.

2. Mathematical formulation

Let $\Omega \subset R^d$, $d \geq 1$ be a bounded, connected, and polyhedral domain. Consider the following parabolic system with the Robin and Neumann boundary conditions

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (\alpha(\mathbf{x})\nabla u) = f(\mathbf{x},t) & \text{in } \Omega \times (0,T), \\ \alpha(\mathbf{x})\dfrac{\partial u}{\partial n} + \gamma(\mathbf{x},t)u(\mathbf{x},t) = g(\mathbf{x},t) & \text{on } \Gamma_i \times (0,T), \\ \alpha(\mathbf{x})\dfrac{\partial u}{\partial n} = q(\mathbf{x},t) & \text{on } \Gamma_c \times (0,T), \\ u(\mathbf{x},0) = 0 & \mathbf{x} \in \Omega. \end{cases}$$

We assume that the boundary $\partial\Omega$ consists of two parts, i.e. $\partial\Omega = \Gamma_i \cup \Gamma_c$, and $\Gamma_c \equiv \Gamma_{c_1} \cup \Gamma_{c_2} \cup \Gamma_{c_3}$ is a finite collection of disjoint, smooth $(d-1)$-dimensional polyhedral domain. Also, $\gamma(\mathbf{x},t)$ and $q(\mathbf{x},t)$ are the heat transfer coefficient (Robin) and heat flux respectively which are contained in the following sets

$$K_1 = \{\gamma(\mathbf{x},t) : 0 < \gamma_1 \leq \gamma(\mathbf{x},t) \leq \gamma_2 < \infty \text{ a.e. in } \Gamma_i \times (0,T)\},$$

$$K_2 = \{q(\mathbf{x},t) : 0 < q_1 \leq q(\mathbf{x},t) \leq q_2 < \infty \text{ a.e. in } \Gamma_c \times (0,T)\},$$

where γ_1, γ_2, q_1, and q_2 are positive given constants. In this problem, the Robin boundary condition is specified on Γ_i and the Neumann boundary condition on Γ_c. We refer readers to [9] and the references therein for more physical backgrounds and [16, 25] for the analytical and numerical methods to solve the inverse problems.

Let $u(\mathbf{x},t)$ solve the forward problem (1) and $z^\delta \in L^2(0,T;L^2(\Gamma_c))$ represent the measured data on Γ_c over $t \in (0,T)$. The parameter δ is used to emphasize the existence of noise in the data. We recall the following lemma for setting the inverse problem to achieve the uniqueness results (see [5]).

proposition 2.1. *Suppose that Ω is an open, bounded, and connected domain with the boundary $\partial\Omega$. The given source strength $f(\mathbf{x},t) \in L^2(0,T;L^2(\Omega))$, $g(\mathbf{x},t) \in$*