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SPECTRAL ELEMENT METHODS ON HYBRID TRIANGULAR AND QUADRILATERAL MESHES

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In memory of Professor Benyu Guo

Abstract. In this paper, we implement and analyse a spectral element method (SEM) on hybrid triangular and quadrilateral element meshes, where the elemental transformation between the triangular element and the reference element is based on the mapping in [17]. We introduce the notion of "quasi-interpolation" to glue the hybrid elements which can build in the singularity of the elemental mapping, and only affects one coefficient of the tensorial nodal basis expansion. Therefore, the hybrid method can be implemented as efficiently as the usual quadrilateral SEM. We also rigorously analyse the "quasi-interpolation" error and the convergence of the hybrid SEM, which show the spectral accuracy can be kept.

Key words. Triangule-rectangle mapping, spectral element method, polygon domain

1. Introduction

The spectral element method, which enjoys both high accuracy of the spectral method and geometric flexibility of the finite element method, has become a powerful tool, perhaps the method of choice, for challenging simulations with stringent accuracy and storage requirement (see, e.g., [19, 5, 13, 2]). The quadrilateral/hexahedral spectral element method (QSEM) has been studied and documented well in literature. We particularly highlight that Guo and Jia [12, 8] conducted a very delicate analysis of the quadrilateral SEM, where the error estimates were featured with the explicit dependence of the geometric parameters of the elements, and where the so-called "quasi-orthogonal projections" played an important part in the analysis. The results therein could provide deep insights into how the quality of the mesh affects the accuracy of spectral element approximations.

It is known that the triangular/tetrahedral spectral element method (TSEM) on unstructured meshes has more flexibility for complex computational domains and adaptivity techniques. Considerable efforts have been devoted to these approaches along the lines: (i) nodal TSEM based on high-order polynomial interpolation on special interpolation points [3, 11, 26]; (ii) modal TSEM based on the Koornwinder-Dubiner (KD) polynomials [14, 6, 13, 15, 21]; and (iii) approximation by non-polynomial functions [23, 16, 4]. It is noteworthy that due to lacking of tensorial structure, these approaches are much more complicated in implementation than QSEM.

One of the main purposes of this work is to further the study of Guo and Jia [7, 8] by considering the scenario when some of the quadrilateral elements deform into triangular elements. Indeed, using hybrid triangular and quadrilateral elements, one can handle more complex domains with more regular meshes, e.g., by tiling the triangular elements along the boundaries of complex obstacles. In practice, one also wishes the implementation of such a hybridisation can inherit the tensorial structure of the QSEM. It is noted that the constants in the error estimates depend on the

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lower bound of $1/J$ (J is the Jacobian of the mapping from a quadrilateral element to the reference square, see [12, (2.9)]). Thus, the constants in the upper bounds become infinity when one of the interior angles is close to π, i.e., the quadrilateral element deforms into a triangular element. This brings about an interesting issue: How to effectively treat singular deformations in implementation without loss of accuracy and rigorously analyse the approach?

The tackle of the issue essentially relies on the triangle-rectangle transformation reported in [17]. The mapping pulls one side (at the middle point) of the triangle to two sides of the rectangle, and results in desirable distributions of the grids, compared with the Duffy mapping [6]. Samson et al [20] proposed a modal approach based on the inspection that the product of any continuous function and $1/J$ is integrable over the reference square, so the singularity of the elemental transformation can be perfectly removed. However, much care is needed for the implementation. Indeed, the nodal basis is more preferable in practice.

In this paper, we introduce the so-called "quasi-interpolation" to glue the neighbouring triangles and rectangles in C^0-sense. Different from the usual tensorial interpolation, this interpolation builds in the "pole" condition of the singular transformation. This however only affects one interpolation coefficients which should be predetermined by some other coefficients. Therefore, we can incorporate this "known" equation in the implementation leading to a minimal amendment of usual QSEM codes. It is noteworthy that this notion is different from the "quasi-orthogonal projection" [12], which was essentially intended to glue and analyse the modal approach for QSEM by separating interior, boundary and vertex modes. We also conduct error analysis of the "quasi-interpolation" and hybrid SEM, and derive the estimates following the spirit of Guo and Jia [12] in terms of showing the explicit dependence of some important parameters.

The rest of the paper is organised as follows. In Section 2, we start with the elemental transformations between quadrilaterals, and the triangle-rectangle mapping. More importantly, we study the situation when one quadrilateral element gradually deforms into a triangular element, and examine how the accuracy is deteriorated and conditioning of the system unpleasantly grows. In Section 3, we introduce the "quasi-interpolation" and derive its interpolation error estimate. In Section 4, we implement the hybrid SEM for elliptic and Stokes problems, conduct some related analysis, and provide various numerical results to show the accuracy.

2. Preliminaries

In this section, we first present the transformation $\boldsymbol{F}_{\Box}$, which transforms the reference square to a convex quadrilateral. Then we test the changes in effect of applying the standard QSEM to elliptic problems when a convex quadrilateral deforms to a triangle gradually. At last, we present the limited transformation $\boldsymbol{F}_{\triangle}$, which transforms the reference square to a triangle, and find out what makes the standard QSEM out of operation when handling the triangular element.

2.1. Elemental transformation between quadrilaterals. Let (ξ, η) be the coordinate system related to the reference square $\Box := \Lambda_\xi \times \Lambda_\eta = (-1,1)^2 = \Lambda^2$. Denote by Q a generic convex quadrilateral with vertices $\big\{Q_j : (x_j, y_j)\big\}_{j=1}^4$ in (x,y)-coordinates. For clarity of presentation, we use boldface letters to denote vectors or vector-valued functions throughout this paper, e.g.,

$$\boldsymbol{x} = (x,y), \quad \boldsymbol{x}_j = (x_j, y_j), \quad \boldsymbol{a}_j = (a_j, b_j), \quad 1 \le j \le 4. \tag{1}$$