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ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS

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In memory of Professor Benyu Guo

Abstract. In this survey article, we present some recent results on the asymptotic behavior of four systems of orthogonal polynomials. These are Stieltjes-Wigert, Hahn, Racah and pseudo-Jacobi polynomials. In each case, the variable z is allowed to be in any part of the complex plane. In some cases, asymptotic formulas are also given for their zeros.

Key words. Asymptotics, orthogonal polynomials, Riemann-Hilbert method, difference-equation techniques.

1. Introduction

Most of the asymptotic results of classical orthogonal polynomials (namely, Hermite $H_n(z)$, Laguerre $L_n^{(\alpha)}(z)$, and Jacobi $P_n^{(\alpha,\beta)}(z)$) can be found in the books of Szegö [45] and Erdélyi et al. [15]. Some of the asymptotic results of a few familiar discrete orthogonal polynomials (e.g., Charlier $C_n(z;a)$, Meixner $M_n(z;\beta,c)$, and Krawtchouk $K_n(z;p,N)$) have been summarized in a survey article of Wong [56]. With the new developments in asymptotic methods based on the Riemann-Hilbert approach [10, 11] and difference equations [48, 50, 51], asymptotic problems of some orthogonal polynomials, which have been considered to be more difficult to tackle, have also been resolved; for instance, discrete Chebyshev [33], Conrad-Flajolet [8], polynomials orthogonal with respect to Freud weights [27, 57], and Tricomi-Carlitz [30, 58]. The reasons for the asymptotic problems of the last few mentioned orthogonal polynomials being more difficult are : (i) they do not satisfy second-order linear ordinary differential equations, and (ii) they do not have integral representations to which one can apply the classical methods of steepest descent or stationary phase.

The purpose of this paper is to give a summary of asymptotic results obtained recently for four systems of orthogonal polynomials; they are Stieltjes-Wigert $S_n(z;q)$, Hahn $Q_n(z;\alpha,\beta,N)$, Racah $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, and pseudo-Jacobi $P_n(z;a,b)$. The problem of finding asymptotic formulas for Stieltjes-Wigert polynomials has been around for some time. However, serious work began only at the beginning of this century. These polynomials do not have integral representations, and neither do they satisfy any second-order linear differential equation. Although they satisfy a three-term recurrence relation [6, p.174], the coefficients of the recurrence relation contain exponentially large terms of the form $q^{-n}, 0 < q < 1$. As a consequence, none of the existing methods for second-order difference equations can be applied. The Hahn polynomial was first introduced by Chebyshev in 1858. Despite its long history, there seems to be no literature on asymptotic results of this polynomial. A difficulty in dealing with asymptotic problems of this polynomial is that it has three free real parameters, whereas the other classical discrete orthogonal polynomials involve only two or less free parameters. This remark also applies to the

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Racah polynomial, which has even one more free parameter. We pick Pseudo-Jacobi polynomials as our last example for presentation, since it also has a long history (over hundred years) and it is not well-known even to the researchers in the field of orthogonal polynomials until just recently. Another reason is that this example illustrates the asymptotic method developed for differential equations with a large parameter [38, Chapter 11], which can be applied to many applied disciplines, and deserves a larger audience.

2. Stieltjes-Wigert polynomials

Let $k > 0$ be a fixed number and

$$q = \exp\{-(2k^2)^{-1}\}. \tag{1}$$

Note that $0 < q < 1$. The q-shifted factorial is given by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j), \qquad n = 1, 2, \cdots .$$

The Stieltjes-Wigert polynomials

$$S_n(z;q) := \sum_{j=0}^{n} \frac{q^{j^2}}{(q;q)_j (q;q)_{n-j}}(-z)^j, \quad n = 0, 1, 2, \cdots , \tag{2}$$

are orthogonal with respect to the weight function

$$w(x) = k\pi^{-\frac{1}{2}} \exp\{-k^2 \log^2 x\} \tag{3}$$

for $0 < x < \infty$; see [26, (3.27.1)] and [39, (18.27.18)]. It is known that these polynomials belong to the indeterminate moment class and the weight function in (3) is not unique; see [7]. By changing the index j to $n-j$ in the explicit expression given in (2), one can easily verify the symmetry relation

$$S_n(z;q) = (-zq^n)^n S_n\left(\frac{1}{zq^{2n}}; q\right). \tag{4}$$

In some literatures, the variable z in (2) is replaced by $q^{\frac{1}{2}}z$; see, for instance, [6], [45] and [52]. The notation for the Stieltjes-Wigert polynomials used in these literatures is

$$p_n(z) = (-1)^n q^{n/2+1/4} \sqrt{(q;q)_n} S_n(q^{\frac{1}{2}}z; q). \tag{5}$$

These polynomials arise in random walks and random matrix formulation of Chern-Simons theory on Seifert manifolds; see [4, 12].

The asymptotics of the Stieltjes-Wigert polynomials, as the degree tends to infinity, has been studied by several authors. First, Wigert [53] in 1923 proved that the polynomials have the limiting behavior

$$\lim_{n\to\infty} (-1)^n q^{-n/2} p_n(z) = \frac{q^{1/4}}{\sqrt{(q;q)_\infty}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k^2+k/2}}{(q;q)_k} z^k, \tag{6}$$

which can be put in terms of the q-Airy function (also known as the Ramanujan function)

$$A_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}(-z)^k. \tag{7}$$