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CONFORMING MIXED TRIANGULAR PRISM ELEMENTS FOR THE LINEAR ELASTICITY PROBLEM

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This paper is dedicated to Professor Ben-yu Guo

Abstract. We propose a family of conforming mixed triangular prism finite elements for solving the classical Hellinger-Reissner mixed problem of the linear elasticity equations in three dimensions. These elements are constructed by product of elements on triangular meshes and elements in one dimension. The well-posedness is established for all elements with $k \geq 1$, which are of $k+1$ order convergence for both the stress and displacement. Besides, a family of reduced stress spaces is proposed by dropping the degrees of polynomial functions associated with faces. As a result, the lowest order conforming mixed triangular prism element has 93 plus 33 degrees of freedom on each element.

Key words. Mixed finite element, triangular prism element, linear elasticity.

1. Introduction

In the Hellinger-Reissner mixed formulation of the linear elasticity equations, it is a challenge to design stable mixed finite element spaces mainly due to the symmetric constraint of the stress tensor, see some earlier work for composite elements and weakly symmetric methods in [2, 6, 7, 30, 34, 35, 36]. In [9], Arnold and Winther designed the first family of mixed finite element methods in two dimensions, based on polynomial shape function spaces. The analogue of the results on tetrahedral meshes can be found in [1, 4], and rectangular and cuboid meshes in [3, 11, 18]. Since the conforming symmetric stress elements have too many degrees of freedom, there are some other methods to overcome this drawback. We refer interested readers to nonconforming mixed elements, see [5, 10, 15, 21, 37] on simplicial meshes, and [26, 31, 39, 40] on rectangular and cuboid meshes. For the weakly symmetric mixed finite element methods for linear elasticity, we also refer to some recent work in [8, 12, 19, 32].

Recently, Hu [23] proposed a family of conforming mixed elements on simplical meshes for any dimension, see [27] and [28] the elements in two and three dimensions, respectively. This new class of elements has fewer degrees of freedom than those in the earlier literature. For $k \geq n$, the stress tensor is discretized by P_{k+1} finite element subspace of $H(\mathrm{div})$ and the displacement by piecewise P_k polynomials. Moreover, a new idea was proposed to analyze the discrete inf-sup condition and the basis functions therein are easy to obtain. For the case that $1 \leq k \leq n-1$, the symmetric tensor spaces are enriched by proper high order $H(\mathrm{div})$ bubble functions to stabilize the discretization in [29]. Another method by stabilization technique to deal with this case can be found in [16]. We also refer to [20] for interior penalty mixed finite element methods by using nonconforming symmetric stress spaces, where the stability is established by introducing the conforming $H(\mathrm{div})$ bubble

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spaces from [23] and nonconforming face-bubble spaces. Corresponding mixed elements on both rectangular and cuboid meshes were constructed in [22], also see [17, 24] for the lowest order mixed elements, while the simplest nonconforming mixed element on n-rectangular meshes can be found in [25].

In this paper, we propose a family of conforming mixed triangular prism elements for the linear elasticity problem. Triangular prism meshes can deal with some columnar regions, and in this case, the triangular prism partition is more easily achieved than the tetrahedral partition. The key idea here of constructing triangular prism elements is using a product structure that each prism can be treated as the product of a triangle and an interval. By dividing the stress variable into three parts, we construct the stress space through a combination of the mixed elasticity element [23, 27] and the Brezzi-Douglas-Marini element [14] on triangular meshes, and some other basic elements in one and two dimensions. In this way, we obtain conforming mixed triangular prism elements for any integer $k \geq 1$. The stability analysis is established by the theory developed in [22, 23, 27, 28, 29]. A family of reduced stress spaces is also proposed by dropping the degree of polynomials associated with faces. The reduced elements still preserve the same order of convergence. The lowest order case has 93 plus 33 degrees of freedom on each element. In addition, by using the lowest order nonconforming mixed element in [10, 21] on triangular meshes, we obtain a nonconforming mixed triangular prism element of first order convergence, of which degrees of freedom are 81 plus 33.

The rest of the paper is organized as follows. In Section 2, we define the conforming mixed triangular prism finite element methods and present the basis functions. In Section 3, we prove the well-posedness of these elements, i.e. the K-ellipticity and the discrete inf-sup condition. By which, the optimal order convergence of the new elements follows. In Section 4, we propose a family of reduced triangular prism elements. In the end, we provide some numerical results.

2. The family of conforming mixed triangular prism elements

Based on the Hellinger-Reissner principle, the linear elasticity problem within a stress-displacement (σ-u) form reads: Find $(\sigma, u) \in \Sigma \times V := H(\mathrm{div}, \Omega; \mathbb{S} := \text{symmetric } \mathbb{R}^{3\times 3}) \times L^2(\Omega; \mathbb{R}^3)$, such that

$$\begin{cases} (A\sigma, \tau) + (\mathrm{div}\tau, u) = 0 & \text{for all } \ \tau \in \Sigma, \\ (\mathrm{div}\sigma, v) = (f, v) & \text{for all } \ v \in V. \end{cases} \tag{1}$$

Here the symmetric tensor space for the stress Σ and the space for the vector displacement V are, respectively,

$$H(\mathrm{div}, \Omega; \mathbb{S}) := \big\{\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \in H(\mathrm{div}, \Omega; \mathbb{R}^{3\times 3}),\ \tau^{\mathrm{T}} = \tau\big\}, \tag{2}$$

$$L^2(\Omega; \mathbb{R}^3) := \{v = (v_1 \ \ v_2 \ \ v_3)^{\mathrm{T}} |\ v_i \in L^2(\Omega; \mathbb{R}), i = 1, 2, 3\}. \tag{3}$$

This paper denotes by $H^k(\omega; X)$ the Sobolev space consisting of functions with domain ω, taking values in the finite-dimensional vector space X, and with all derivatives of order at most k square-integrable. For our purposes, the range space X will be either $\mathbb{S}$, $\mathbb{R}^3$, $\mathbb{R}^2$, or $\mathbb{R}$, and in some cases, X will be $\mathbb{S}_2 :=$ symmetric $\mathbb{R}^{2\times 2}$ as well. Let $\|\cdot\|_{k,\omega}$ be the norm of $H^k(\omega)$ and $H(\mathrm{div}, \omega; \mathbb{S})$ consist of square-integrable symmetric matrix fields with square-integrable divergence. The $H(\mathrm{div})$ norm is defined by

$$\|\tau\|^2_{H(\mathrm{div},\omega)} := \|\tau\|^2_{0,\omega} + \|\mathrm{div}\tau\|^2_{0,\omega}.$$