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# PRECONDITIONING TECHNIQUES IN CHEBYSHEV COLLOCATION METHOD FOR ELLIPTIC EQUATIONS

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*This paper is dedicated to memory of late Professor Benyu Guo*

**Abstract.** When one approximates elliptic equations by the spectral collocation method on the Chebyshev-Gauss-Lobatto (CGL) grid, the resulting coefficient matrix is dense and ill-conditioned. It is known that a good preconditioner, in the sense that the preconditioned system becomes well conditioned, can be constructed with finite difference on the CGL grid. However, there is a lack of an efficient solver for this preconditioner in multi-dimension. A modified preconditioner based on the approximate inverse technique is constructed in this paper. The computational cost of each iteration in solving the preconditioned system is $\mathcal{O}(\ell N_x N_y \log N_x)$, where $N_x, N_y$ are the grid sizes in each direction and $\ell$ is a small integer. Numerical examples are given to demonstrate the efficiency of the proposed preconditioner.

**Key words.** Chebyshev collocation method, elliptic equation, finite-difference preconditioner, approximate inverse.

## 1. Introduction

We consider a two-dimensional separable elliptic equation

$$(1)\quad -\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(b(y)\frac{\partial u}{\partial y}\right)+c(x)d(y)u(x,y)=f(x,y)\quad \text{in } \Omega=(-1,1)^2$$

with homogeneous Dirichlet boundary conditions

$$u = 0 \text{ on } \partial\Omega,$$

where the coefficient functions $a(x)$, $b(y)$, $c(x)$, $d(y)$ and $f(x,y)$ are continuous, and $0<\alpha\le a(x), b(y)\le\beta$ in $\Omega$ for some positive constants $\alpha$ and $\beta$, and $c(x)d(y)\ge 0$.

A very efficient accurate method for obtaining approximate solution of the above boundary-value problem is the Chebyshev collocation method [3, 4, 7, 9, 11, 12], which uses the Lagrange nodal basis functions based on the Chebyshev collocation points. However, due to the global nature of the Lagrange basis polynomials, the associated linear systems are dense and ill-conditioned. Thus it becomes prohibitive to use a direct inversion method or an iterative method without preconditioning in the multi-dimensional case, so it is imperative to use an iterative method with a good preconditioner.

Finite element/finite difference preconditioners have been widely used since the original work by Orszag [9]. Haldenwang et al. [5] proved that the finite difference method based on the Chebyshev collocation points in the one-dimensional case leads to a good preconditioner. The properties of the finite element/finite difference preconditioners in the two-dimensional case were rigorously established by Kim and Parter [6, 7]. Thus, the application of the Krylov subspace methods [1], such as generalized minimal residual method (GMRES), leads to an iterative solver converging to the algebraic solution within a constant number of steps that depends on the required accuracy, but not on the number of unknowns.

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However, such a preconditioned method requires solving the preconditioner system, i.e., solving the finite element/finite difference system on the spectral collocation points. How to efficiently apply the preconditioners is a challenging problem since the grid formed by spectral collocation points, containing long-thin elements, is not shape-regular. We note that Shen et al. [13] developed a finite element multigrid preconditioner for the second-order elliptic equations. In this paper, we seek to develop an approximate preconditioner by exploring the algebraic properties of the finite difference preconditioner.

It is obvious that the two-dimensional finite difference preconditioner is a nonsymmetric block tridiagonal matrix. Approximating this matrix to construct a new efficient preconditioner is a natural idea. In [8], Ng and Pan proposed an approximate inverse method to modify circulant-plus-diagonal preconditioners for solving Toeplitz-plus-diagonal systems. Their idea is to use circulant matrices to approximate the inversion of Toeplitz matrices and then combine the rows of these matrices together. As the resulting preconditioner is already of the inverted form, only matrix-vector multiplications are required in the preconditioning step. Recently, Pan et al. [10] also proposed approximate inverse preconditioners for diagonal-times-Toeplitz matrices.

The main purpose of this paper is to propose and develop approximate inverse preconditioners for two-dimensional elliptic operators, based on the modification of the finite-difference operator discretized on the CGL grid. First, we use a scaling strategy to approximate the finite-difference operator. Then we construct an approximate inverse preconditioner to approximate the inverse of scaled Laplacian-plus-diagonal matrices and combine them together row-by-row. In order to reduce the influence of the various coefficients, an interpolation method with the eigenvalues of Laplacian is utilized. Special interpolation nodes are chosen to improve the accuracy of approximation. By use of the discrete sine transform (DST), the resulting preconditioner can be efficiently implemented with $\mathcal{O}(\ell N_x N_y \log N_x)$ operations, where the small integer $\ell$ is independent of $N_x$ and $N_y$. Numerical examples are given to demonstrate the effectiveness of the proposed preconditioner.

The paper is organized as follows. In Section 2, we introduce the Chebyshev collocation method for the elliptic operator and the associated finite-difference operator. In Section 3, we construct the proposed preconditioners. Numerical examples are given to demonstrate the performance of the proposed preconditioner in Section 4. In the final section, concluding remarks are given.

## 2. The Chebyshev-collocation and the finite-difference operator

In this section we recall the Chebyshev-collocation method for the elliptic operator and the associated finite-difference operator. Let $\mathcal{P}_N$ be the space of polynomials of degree less than or equal to $N$. Let

$$x_j = -\cos\left(\frac{j\pi}{N}\right), \qquad j = 0, 1, \ldots, N,$$

which are the CGL points.

**2.1. The one-dimensional case.** Consider the one-dimensional elliptic problems

$$-(a(x)u'(x))' + c(x)u(x) = f(x), \qquad x \in (-1,1); \quad u(\pm 1) = 0. \tag{2}$$

The Chebyshev-collocation method for (2) is to find $u_N \in \mathcal{X}_N := \{v \in \mathcal{P}_N : v(\pm 1) = 0\}$ such that

$$-(au_N')'|_{x=x_k} + c(x_k)u_N(x_k) = f(x_k), \quad k = 1, 2, \ldots, N-1. \tag{3}$$