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SUPERCONVERGENCE OF A QUADRATIC FINITE ELEMENT METHOD ON ADAPTIVELY REFINED ANISOTROPIC MESHES

WEIMING CAO

This paper is dedicated to the memory of Professor Benyu Guo

Abstract. We establish in this paper the supercloseness of the quadratic finite element solution of a two dimensional elliptic problem to the piecewise quadratic interpolation of its exact solution. The assumption is that the partition of the solution domain is quasi-uniform under a Riemannian metric and that each pair of the adjacent elements in the partition forms an approximate parallelogram. This result extends our previous one in [7] for the linear finite element approximations based on adaptively refined anisotropic meshes. It also generalizes the results by Huang and Xu in [13] for the supercloseness of the quadratic elements based on the mildly structured quasi-uniform meshes. A distinct feature of our analysis is that we transform the error estimates on each physical element to that on an equilateral standard element, and then focus on the algebraic properties of the Jacobians of the affine mappings from the standard element to the physical elements. We believe this idea is also useful for the superconvergence study of other types of elements on unstructured meshes.

Key words. Quadratic elements, superconvergence, anisotropic meshes.

1. Introduction

Superconvergence study in finite element approximations has been an area of research for several decades. Classical superconvergence analysis is mostly performed on approximations based on uniform meshes or structured meshes, since superconvergence is generally the result of cancellation of certain lower order terms in the discretization, which relies on the local symmetry of the partition of the solution domains, [2, 22, 26, 27]. There have been much recent developments in extending the study to the FEM based on general types of meshes, see, e.g., [3, 12, 15, 24, 25]. Bank and Xu [3] and Huang and Xu [13] introduced a number of basic identities which involves explicitly the geometric properties of the elements in the partition, and established the supercloseness of the linear and quadratic finite element solution of an two dimensional elliptic equation to the interpolation of its exact solution on general mildly structure quasi-uniform meshes.

For practical applications of the finite element method, the partition of the domain are often adaptively refined, and the meshes are no longer quasi-uniform or even shape regular. In this case, superconvergence is often still observed. Various error estimators have been designed based on such a property to guide the mesh refinement process, see, e.g., [10, 14, 16, 17, 20, 21]. Therefore, understanding of the superconvergence on adaptively refined unstructured meshes may offer useful insights to the practitioners of the finite element method. There have been some recent efforts in this area of study e.g., Wu and Zhang [23] established the superconvergence for linear finite element approximation of a singular perturbed problem based on a pre-defined graded mesh. However, the theoretical work in this area is still limited due to the technical complexity involved in deriving the expressions for discretization errors in general element geometries.

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By using the notion of quasi-uniform meshes under a Riemannian metric [9, 5, 6], we extended in [7] the superconvergence analysis for linear finite element approximations on mildly structure quasi-uniform meshes in [3] to certain adaptively refined anisotropic meshes. An innovation for the analysis in [7] is the development of the notion of approximate parallelogram for anisotropic meshes. Based on this superconvergence result, we established in [8] the effectiveness of several commonly used gradient recovery type error estimators for the FEM based on adaptively refined anisotropic meshes.

In this paper, we extend our analysis in [7] for linear FE approximations to a quadratic FE approximation on anisotropic meshes. We establish rigorously the supercloseness of the finite element solution of a two dimensional elliptic equation to the piecewise quadratic interpolation of the exact solution. This conclusion also generalizes the results by Huang and Xu [13] for the supercloseness of quadratic elements on the mildly structured quasi-uniform meshes As is documented in the study of a Laplace equation in [13], superconvergence analysis for quadratic elements on general meshes is much more complicated technically than for linear elements, and considering anisotropic features of the partition makes the study even more difficult. We simplify the task by performing the analysis on the standard element. More specifically, we select the equilateral triangle with vertices on the unit circle as the standard element, and transform the discretization errors on each physical element to the standard element. Then all the estimates involving the geometric properties of the physical elements become those involving the algebraic properties of the Jacobians of the affine mappings from the standard element to the physical elements. This approach was first used in our superconvergence study for linear elements in [7]. It made much easier technically the derivation of various error bounds needed for the analysis. We believe this idea is useful for the superconvergence analysis in other types of problems.

An outline of this paper is as follows: We describe in Section 2 the model problem, the anisotropic partitions of the solution domain, and the measure of the anisotropic features of the higher order derivatives of solutions. We list in Section 3 a number of basic lemmas and then establish the supercloseness of the quadratic finite element solution to the piecewise quadratic interpolation of the exact solution. We provide in Section 4 two numerical examples and finish the paper with some discussions in Section 5.

2. FE Approximation Based on Anisotropic Meshes

We consider the following homogeneous Dirichlet problem of a second order elliptic equation:

$$\begin{cases} -\nabla\cdot(A\nabla u+\mathbf{b}\ u)+d\ u=f, & \text{in } \Omega \\ u|_{\partial\Omega}=0, \end{cases} \tag{1}$$

where A is symmetric positive definite (SPD) constant matrix, and $\mathbf{b}$, d, and f are suitably smooth functions. Furthermore, (1) is assumed to be strongly elliptic.

FE approximation: Let $\{\mathcal{T}_N\}$ be a family of triangulations of Ω satisfying the basic requirement that the intersection of the closures of any two elements is either the empty set, a point, or an entire edge. Here N stands for the total number of elements in $\mathcal{T}_N$. We use N, instead of the usual element diameter h to characterize the fineness of the partition, because in anisotropic meshes an element may have very different length scales in different directions. Define S_N be the space of continuous piecewise quadratic polynomials over partition $\mathcal{T}_N$, and $V_N =$