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# REGULARIZATION AND ROTHE DISCRETIZATION OF SEMI-EXPLICIT OPERATOR DAES

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**Abstract.** A general framework for the regularization of constrained PDEs, also called operator differential-algebraic equations (operator DAEs), is presented. The given procedure works for semi-explicit and semi-linear operator DAEs of first order including the Navier-Stokes and other flow equations. The proposed reformulation is consistent, i.e., the solution of the PDE remains untouched. Its main advantage is that it regularizes the operator DAE in the sense that a semi-discretization in space leads to a DAE of lower index. Furthermore, a stability analysis is presented for the linear case, which shows that the regularization provides benefits also for the application of the Rothe method. For this, the influence of perturbations is analyzed for the different formulations. The results are verified by means of a numerical example with an adaptive space discretization.

**Key words.** PDAE, operator DAE, regularization, index reduction, Rothe method, method of lines, perturbation analysis

## 1. Introduction

Constrained PDEs arise naturally in the modelling of physical, chemical, and many other real-world phenomena. They occur whenever different PDE models are coupled, e.g., via mutual variables at the interfaces, since the coupling is typically modelled via algebraic constraints. Such models are widely used in flexible multi-body dynamics, e.g., the pantograph and catenary benchmark problem [6] or the flexible slider crank mechanism [30, 31]. Also flow equations such as the Navier-Stokes equations [33, 35] can be seen as constrained PDEs due to the coupling of momentum equation to the divergence-free constraint. Further applications can be found in circuit simulation [34], electromagnetics, and chemical engineering [9].

We consider these equation systems of ordinary or partial differential equations (ODEs, PDEs) and algebraic equations in line with other constrained PDEs – often referred to as PDAEs – as differential-algebraic equations (DAEs) in function spaces, so-called *abstract* or *operator DAEs*.

Despite the large range of applications and the advantages from the modeling perspective, the mathematical analysis of operator DAEs is full of open research questions. There is still no common classification like the index concepts for DAEs [21, Ch. 12]. The generalization of the *tractability index* as proposed, e.g., in [34] does not apply for the commonly used formulation by means of Gelfand triples. The very general concept of the *perturbation index*, as it was defined in [25] for linear PDAEs, applies under strong regularity conditions but is still ambiguous in the choice of the norm in which one measures the perturbation and their derivatives. Also the *differentiation index* was generalized to PDAEs [22] but has difficulties with the agreement of the PDAE index with the index of the semi-discretized DAE. Yet another idea is to classify the index of a PDAE directly by the index that may

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be determined after a spatial discretization. This, however, leads to the similar unclear problem, what a good discretization of a PDAE is.

Within this paper, we analyse constrained systems of first order and semi-explicit structure. Particularly, we consider systems of the form

$$\dot{u}(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t), \tag{1a}$$

$$\mathcal{B}u(t) = \mathcal{G}(t). \tag{1b}$$

Therein, $\lambda$ denotes the Lagrange multiplier, which enforces the linear constraint $\mathcal{B}u = \mathcal{G}$. In view of numerical simulations, the incorporation of the constraints via a Lagrange multiplier and a suitable reformulation seem promising and follow the paradigm in the treatment of DAEs, that it is preferable to collect all available information in the form of constraints instead of eliminating them. For the Navier-Stokes equations this means to maintain the pressure as part of the system.

In this paper, we introduce a regularization or *index reduction method* for the PDAE (1) without introducing an index as such. We rather refer to the well-defined index of the semi-discrete system after a spatial discretization by mixed finite elements. In other words, we propose a reformulation of the given PDAE system such that a semi-discretization leads to a DAE of lower index.

A transformation on operator level can be the base for numerically advantageous discretization schemes. The commonly taken approach of first discretizing and then transforming the equations comes with the latent risk that the algebraic manipulation are not valid in infinite dimensions [17]. This may cause instabilities or inconsistencies as the discretization becomes more accurate. The taken approach has been introduced for second-order systems appearing in elastodynamics [2] and for flow equations [4] before. Here, we consider the more general case with time-dependent constraints and provide the functional analytical framework.

The main contribution of this paper is then the analysis of the Rothe discretization through the application of the implicit Euler scheme to the operator DAE (1). As for the finite-dimensional case, we expect a different behavior of the variables $u$ and $\lambda$. It will turn out that we need stronger regularity assumptions to prove the convergence of the Lagrange multiplier. Among others, we consider the influence of perturbations and quantify them in a general convergence result. We show that the proposed reformulation improves the robustness against such perturbations, as we confirm numerically for a simulation setup with adaptive, and thus changing, meshes.

The paper is organized as follows. In Section 2 we provide the theoretical framework for the formulation of operator differential equations. These tools are then used for the formulation and regularization of the operator DAEs in Section 3 in which we also analyse the influence of perturbations. The advantages of the obtained formulation is topic of Section 4. We consider the discretization in time, which corresponds to the Rothe method for time-dependent PDEs in Section 5. Further, we prove the convergence of the implicit Euler scheme and discuss the resulting advantages in terms of perturbations. Finally, we illustrate the obtained theoretical results in a numerical simulation of the Navier-Stokes equations in Section 6 and conclude the paper in Section 7.