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# AN ENSEMBLE-BASED CONVENTIONAL TURBULENCE MODEL FOR FLUID-FLUID INTERACTION

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*Dedicated to Professor William J. Layton on the occasion of his* $60^{th}$ *birthday*

**Abstract.** A statistical turbulence model is proposed for ensemble calculations with two fluids coupled across a flat interface, motivated by atmosphere-ocean interaction. For applications, like climate research, the response of an equilibrium climate state to variations in forcings is important to interrogate predictive capabilities of simulations. The method proposed here focuses on the computation of the ensemble mean-flow fluid velocities. In particular, a closure model is used for the Reynolds stresses that accounts for the fluid behavior at the interface. The model is shown to converge at long times to statistical equilibrium and an analogous, discrete result is shown for two numerical methods. Some matrix assembly costs are reduced with this approach. Computations are performed with monolithic (implicit) and partitioned coupling of the fluid velocities; the former being too expensive for practical computing, but providing a point of comparison to see the effect of partitioning on the ensemble statistics. It is observed that the partitioned methods reproduce the mean-flow behavior well, but may introduce some long-time statistical bias.

**Key words.** Atmosphere-ocean, variance, statistical equilibrium

## 1. Introduction

Ensemble calculations with global circulation models (GCMs) are an important component of climate variability studies. Many long-time integrations are performed to assess the average near-surface temperature change in response to changes in forcings for the climate system at radiative equilibrium. There are multiple sources of uncertainty in calculated responses. For a given computational model, some studies focus on parametric uncertainty (see [8], 9.2.2.2). One way to mitigate cost is to apply statistical models for simulation responses that require only a modest number of uncertain parameters and ensemble members, for example in perturbed physics ensemble methods (*e.g.* [22]). Given a moderate number of ensembles, this paper investigates another possible cost-reduction measure. Based on the work of Jiang, Kaya and Layton [10], a method is proposed herein to compute ensemble-mean flow states efficiently, by using a conventional (statistical) turbulence model (CTM). CTM models (like RANS [1] or $k-\epsilon$ [20]) seek to reduce the number of degrees of freedom required to resolve mean fluid behavior. The method has the additional benefit of reducing some matrix-assembly costs for the ensemble computations.

The focus is on atmosphere-ocean interaction (AOI). A key aspect of many AOI models is to avoid resolving the boundary layers, instead relying on boundary conditions that conserve fluxes across the layers. In order to reduce the problem down but retain this key mathematical detail, Connors, Howell and Layton investigated a model of two incompressible fluids coupled across a flat interface [5]. A similar model is adopted here, since it provides a convenient setting in which to incorporate

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the work of Jiang, Kaya and Layton. However, it is necessary to account for the special dynamics in AOI introduced by the differences in vertical versus horizontal scaling, and also by the interface boundary conditions.

**1.1. A conventional turbulence model for coupled fluids.** Consider an ensemble of velocities and pressures for flows in the atmosphere and ocean (domains $\Omega_{\mathcal{A}}, \Omega_{\mathcal{O}} \subset \mathbb{R}^d$, $d = 2, 3$, respectively), satisfying

$$\partial_t \mathbf{u}_j - \mathcal{D}_{\mathcal{A}}(\mathbf{u}_j) + \mathbf{u}_j \cdot \nabla \mathbf{u}_j + \nabla p_j = \mathbf{f}_{\mathcal{A}j} \text{ on } \Omega_{\mathcal{A}} \times (0, T], \tag{1}$$

$$\nabla \cdot \mathbf{u}_j = 0 \text{ on } \Omega_{\mathcal{A}} \times (0, T], \tag{2}$$

$$\mathbf{u}_j(\mathbf{x}, t = 0) = \mathbf{u}_j^0(\mathbf{x}) \text{ on } \Omega_{\mathcal{A}}, \tag{3}$$

$$\partial_t \mathbf{v}_j - \mathcal{D}_{\mathcal{O}}(\mathbf{v}_j) + \mathbf{v}_j \cdot \nabla \mathbf{v}_j + \nabla q_j = \mathbf{f}_{\mathcal{O}j} \text{ on } \Omega_{\mathcal{O}} \times (0, T], \tag{4}$$

$$\nabla \cdot \mathbf{v}_j = 0 \text{ on } \Omega_{\mathcal{O}} \times (0, T], \tag{5}$$

$$\mathbf{v}_j(\mathbf{x}, t = 0) = \mathbf{v}_j^0(\mathbf{x}) \text{ on } \Omega_{\mathcal{O}}. \tag{6}$$

Here, $\mathcal{D}_{\mathcal{A}}(\mathbf{u}_j)$ and $\mathcal{D}_{\mathcal{O}}(\mathbf{v}_j)$ are viscosity terms. Let $\mathbf{D}(\mathbf{u})$ represent any $d \times d$ tensor or matrix, with entries $\mathbf{D}(\mathbf{u})_{ij}$, $1 \le i, j \le d$. Define a decomposition by

$$\begin{aligned}
&\mathbf{D}(\mathbf{u}) = \mathbf{D}(\mathbf{u})^H + \mathbf{D}(\mathbf{u})^\perp, \\
&\left(\mathbf{D}(\mathbf{u})^H\right)_{ij} = \begin{cases} (\mathbf{D}(\mathbf{u}))_{ij} & \text{for } i = 1, \dots, d-1,\ j = 1, \dots, d, \\ 0, & \text{otherwise} \end{cases} \\
&\left(\mathbf{D}(\mathbf{u})^\perp\right)_{ij} = \begin{cases} (\mathbf{D}(\mathbf{u}))_{ij} & \text{for } i = d,\ j = 1, \dots, d, \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{7}$$

Now let $\mathbf{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ be specifically the viscous part of the Cauchy stress tensor. The diffusion terms are decomposed into horizontal and vertical terms:

$$\mathcal{D}_{\mathcal{A}}(\mathbf{u}_j) = 2\nabla \cdot \left( {\nu_{\mathcal{A}}}^H \mathbf{D}(\mathbf{u}_j)^H + {\nu_{\mathcal{A}}}^\perp \mathbf{D}(\mathbf{u}_j)^\perp \right), \tag{8}$$

$$\mathcal{D}_{\mathcal{O}}(\mathbf{v}_j) = 2\nabla \cdot \left( {\nu_{\mathcal{O}}}^H \mathbf{D}(\mathbf{v}_j)^H + {\nu_{\mathcal{O}}}^\perp \mathbf{D}(\mathbf{v}_j)^\perp \right). \tag{9}$$

The constants ${\nu_{\mathcal{A}}}^H > 0$ and ${\nu_{\mathcal{O}}}^H > 0$ are horizontal diffusion parameters, whereas ${\nu_{\mathcal{A}}}^\perp > 0$ and ${\nu_{\mathcal{O}}}^\perp > 0$ are (constant) vertical diffusion parameters. The horizontal scale is much larger than the vertical scale for atmosphere-ocean simulations. Due to the nature of flow features that result from this scale discrepancy, it is typical in practice to treat horizontal and vertical diffusion processes differently (see, *e.g.* [21, 23]). While many other aspects of typical atmosphere-ocean models are not included here, the above model retains the necessary mathematical features for the investigation in this paper. Boundary conditions are discussed later.

We assume that $j = 1, \dots, J$ and define the ensemble average, say $a$, of any collection of $J$ objects $a_j$ by

$$a \equiv < a_j > \equiv \frac{1}{J} \sum_{j=1}^{J} a_j. \tag{10}$$

A model for the ensemble-averaged mean flow is

$$\partial_t \mathbf{u} - \mathcal{D}_{\mathcal{A}}(\mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nabla \cdot R(\mathbf{u}, \mathbf{u}) = \mathbf{f}_{\mathcal{A}} \text{ on } \Omega_{\mathcal{A}} \times (0, T], \tag{11}$$

$$\nabla \cdot \mathbf{u} = 0 \text{ on } \Omega_{\mathcal{A}} \times (0, T], \tag{12}$$

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x}) \text{ on } \Omega_{\mathcal{A}}, \tag{13}$$

$$\partial_t \mathbf{v} - \mathcal{D}_{\mathcal{O}}(\mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla q - \nabla \cdot R(\mathbf{v}, \mathbf{v}) = \mathbf{f}_{\mathcal{O}} \text{ on } \Omega_{\mathcal{O}} \times (0, T], \tag{14}$$

$$\nabla \cdot \mathbf{v} = 0 \text{ on } \Omega_{\mathcal{O}} \times (0, T], \tag{15}$$

$$\mathbf{v}(\mathbf{x}, t = 0) = \mathbf{v}_0(\mathbf{x}) \text{ on } \Omega_{\mathcal{O}}, \tag{16}$$