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A GLIMPSE ON FOURIER ANALYSIS: THIRD STAGE

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Dedicated to Professor William J. Layton on the occasion of his 60th birthday

Abstract. The third stage of Fourier analysis is considered herein. A generalized Fourier series is considered with real valued, locally integrable functions.

Key words. Fourier analysis, third stage

The **third** stage of Fourier Analysis is concerned with generalized Fourier series of the form

$$\sum_{k=1}^{\infty} a_k \exp\{i f_k(t)\}, \tag{1}$$

in which $a_k \in C$, $k \geq 1$, while $f_k(t) : R \to R$, $k \geq 1$, are real valued functions, at least locally integrable on R: $f_k \in L^1_{loc}(R, R)$.

The *first* stage and the *second* correspond to the choice of linear $f_k(t) = \lambda_k t$, $\lambda_k \in R$, leading to the periodic functions when $\lambda_k = k\omega$, $\omega > 0$, $k \geq 0$, and to the Bohr *almost periodic* functions when $\lambda_k \in R$ are arbitrary.

Only for nonlinear $f_k(t)$ one can obtain generalized Fourier series characterizing oscillatory functions, of a more general nature than those in the first of second stages.

A tool helping us to construct series like (1) is the *Poincaré mean value* of a function, on the real line R. The formula used by Poincaré (*Nouvelles Méthodes de la Mécanique Céleste*, 1892-3) is

$$M\{f\} = \lim_{x \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \mathrm{dt}, \tag{2}$$

with $f : R - C$ a locally integrable function for which the limit exists.

All classes/spaces of almost periodic functions (Bohr, Stepanov, Besicovitch) consist of elements for which the mean value in (2) exists (finite!).

The following formula, as noticed by Poincare, is valued for $\lambda \in R$:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp\{i\lambda t\} \mathrm{dt} = \begin{cases} 1, & \lambda = 0 \\ 0, & \lambda \neq 0. \end{cases} \tag{3}$$

Formula (3) is sort of an orthogonality condition, since it implies

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp\{(\lambda_k - \lambda_j)t\} \mathrm{dt} = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases} \tag{4}$$

where $\{\lambda_k : k \geq 1\} \subset R$ is a sequence with distinct terms.

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In order to construct series like (1), it appears possible to obtain solutions, if any, of the functional equation in $\lambda(t)$,

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\exp\{i\lambda(t)\}\mathrm{dt} = \begin{cases}1, & \lambda(t)=0\\ 0, & \lambda(t)\neq 0.\end{cases} \tag{5}$$

with $\lambda(t)$ real valued and locally integrable on R.

We have, so far, examples of function classes/spaces providing solutions to (5), infinitely many. The *first* space appears to be due to V.F. Osipov, in the book *Almost Periodic Functions of Bohr-Fresnel* (Russian), University of Sankt Petersburg Press, 1992, who has constructed such a space, in which case

$$\lambda_k(t) = \alpha t^2 + \mu t$$

$\alpha =$ const. $\in R$, $k \geq 1$. Osipov's construction, according to his statement, has been inspired by a seminal paper of N. Wiener (Acta Mathematica, vol. 55, 1930), to whom the Fresnel waves, $w(t) = \exp\{i(\alpha t^2 + \mu t)\}$, are attributed. Using these waves, Osipov constructed this space, called by him the space of a.p. functions of Bohr-Fresnel.

The functions in this space, obviously of oscillatory type, correspond to generalized Fourier series of the form

$$\sum_{k=1}^{\infty} a_k \exp\{i(\alpha t^2 + \lambda_k t)\}, \tag{6}$$

with α depending on the function to be represented by (6) a real number and $\lambda_k \in R$, $k \geq 1$, distinct.

The Parseval equation holds

$$\sum_{k=1}^{\infty}|f_k|^2 = \lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}|f(t)|^2\mathrm{dt}, \tag{7}$$

where

$$f(t) \sim \sum_{k=1}^{\infty} f_k \exp\{i(\alpha t^2 + \lambda_k t)\}, \tag{8}$$

and

$$f_k = \lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell} f(t)\exp\{-i(\alpha t^2 + \lambda_k t)\}\mathrm{dt}, \quad k \geq 1, \tag{9}$$

Other properties for the Bohr-Fresnel functions hold true, similar to those encountered for the Bohr almost periodic function: an example is the approximation (uniformly on R) of these functions by generalized trigonometric polynomials with exponents in the class of functions of the form $\alpha t^2 + \mu t$, $\mu \in R$ (each taking a finite number of values).

The *second* space of generalized oscillatory functions has been constructed by Ch. Zhang (J. Fourier Analysis, vol. 12(2006); also IEEE Trans. AC, vol. 49(2004)). The construction is reproduced in one of our papers [3] and relies on the properties of a function algebra whose element are generalized polynomials of the form

$$\lambda_k(t) = \sum_{j=1}^{k} c_j t^{\alpha_j}, \quad t \geq 0, \tag{10}$$

where $c_j \in C$, $j = 1, 2, ..., k$, and $\lambda_k(t)$ of the form (10), $c_j \in \mathbb{C}$, $\alpha_1 > ... > \alpha_k > 0$, $k \geq 1$.